Def: A matrix is totally positive (resp., non-negative) if all of its minors are positive (resp., non-negative) real numbers.

1930's: Systematic study of these matrices by Schoenberg, Gautschi, Krein, Whitney. Since then, this field has been linked to:
- Oscillations in mechanical systems
- Stochastic processes
- Planar resistor networks...

1994: Lusztig found a surprising connection between total positivity and canonical bases in quantum groups. This led to his introduction of the totally positive and totally nonnegative parts $G_{zt}, G_{zn}$ in every real reductive group. Similarly, he introduced the totally pos. and non-neg. parts of any generalized partial flag variety $G/P$.

1996-2001: Fomin $+$ Zelevinsky, also Berenstein-Fomin-Zelevinsky further developed Lusztig's theory of total positivity in $G$ and tried to understand Lusztig's dual canonical basis in a "concrete" way. This work led to the introduction of cluster algebras by Fomin $+$ Zelevinsky in 2002.
Today: Explain how the study of total positivity in $G \rightarrow \text{cluster algebras}$. We'll look at the case $G = SL_r$, where $G_{>0}$ and $G_{\geq 0}$ recover totally positive and non-negative matrices (w/ determinant $I$).

Questions one might ask:

1. How can we parameterize the set of all elements in $G_{>0}$? in $G_{\geq 0}$?
2. How many minors must we test to deduce that a matrix $M \in G_{>0}$? Which minors?

1. There is a general procedure for producing totally nonnegative matrices. Fix a planar network - an acyclic directed planar graph $\Gamma$ whose edges have weights.

The weight of a directed path in $\Gamma$ is defined to be the product of the weights of the edges. The weight matrix $X(\Gamma)$ is an $n \times n$ matrix $(a_{ij})$ where

$$a_{ij} = \text{sum of all weights from } i \text{ to } j.$$ 

Here,

$$X(\Gamma) = \begin{pmatrix} d & dh & dhi \\ bd & bdh+e & bdh+eg+e \\ abd & abdh+ae+ce & abdh+(ac+e)(g+i)+f \end{pmatrix}$$
Lemma (Lindstrom-Gessel-Viennot): All minors of such a matrix are polynomials in the edge weight with positive coefficients. (There is a combinatorial interpretation for $\Delta_{I,J}$ as the sum of weights of all vertex-disjoint paths from the source $I$ to the sink $J$.)

So if each edge weight is in $\mathbb{R}_{\geq 0}$, $x(\mathbf{r})$ is totally nonnegative.

Moreover (A. Whitney ’52, Fomin, Zelevinsky) The map $(\mathbb{R}_{\geq 0})^{9} \rightarrow 3 \times 3$ matrices given by $(a,b,c,\ldots, i) \mapsto x(\mathbf{r})$ is a bijection from $(\mathbb{R}_{\geq 0})^{9}$ to totally positive $3 \times 3$ matrices.

(And the obvious generalization works for $n \times n$ matrices.)

Planar networks are a useful tool for parameterizing totally positive matrices and related varieties.

On question 2: (How many, & which minors do we need to test if a matrix is TP?)

Double wiring diagrams (Fomin + Zelevinsky) Choose two families of piecewise straight lines, each family colored w/ one of two colors, s.t. each pair of lines of like colors intersect exactly once.
Remark: if we look at the set of lines in a fixed color, this encodes a reduced decomposition for the longest permutation $\omega_0 = (n, n-1, \ldots, 2, 1)$.

Assign to each chamber of a diagram a pair of subsets of the set $[1,n] = \{1, \ldots, n\}$: each subset indicates which lines of the corresponding color pass below the chamber:

Theorem (Fomin+Zelevinsky): Each double wiring diagram — each of which is determined by a shuffle of two reduced decomps for $\omega_0$ — gives rise to the following criterion: an $n \times n$ matrix is totally positive iff all its chamber minors are positive.

Example above says: A $3 \times 3$ matrix $M$ is totally positive iff the following minors are positive:

$$
\begin{align*}
\Delta_{12,123}(M) & \Delta_{13,12}(M) & \Delta_{12,23}(M) & \Delta_{12,123}(M) \\
\Delta_{3,12}(M) & \Delta_{2,12}(M) & \Delta_{12,1}(M) & \Delta_{1,3}(M)
\end{align*}
$$

We get a lot of TP criteria this way. Let’s make a chart showing all of them.
Here is an "exchange graph" showing TP criteria for $GL_3$. I've drawn in the degree of each vertex.

Notation:

- $a = x_{11}$, $A = \Delta_{23,23}$
- $b = x_{12}$, $B = \Delta_{23,13}$
- $c = x_{21}$, $C = \Delta_{13,23}$
- $d = x_{22}$, $D = \Delta_{13,13}$
- $e = x_{23}$, $E = \Delta_{13,12}$
- $f = x_{32}$, $F = \Delta_{12,13}$
- $g = x_{33}$, $G = \Delta_{12,12}$

Minors corresponding to unbounded chambers (appear in all criteria):

- $x_{13}$
- $x_{31}$
- $\Delta_{12,23}$
- $\Delta_{23,12}$
- $\det(x)$

This is the TP criteria we saw earlier, using the double wiring diagram. analogus to "frozen" Coefficient Variables

Figure 8. Total positivity criteria for $GL_3$

Two arrangements Arr(i) and Arr(i') whose isotopy types are adjacent in the graph

Ex: For each edge in this graph, find an algebraic relation that relates the variables of the 2 corresponding "clusters"
Fomin & Zelevinsky realized that perhaps they were just looking at a piece of a bigger graph—that there should be some "mutation" procedure to go from each TP criteria ("cluster") to 4 others.

In this example ($SL_3$—which is basically the same as $GL_3$), there are actually 50 clusters, so we were missing 16 before.

It is of type $D_n$.

The "coefficient" variables are $x_{13}, x_{31}, \Delta_{12, 23}, \Delta_{23, 12}$, and the cluster variables are:

(i) the other 19 minors of a $3 \times 3$ matrix (except det)
(ii) $X_{12} X_{21} X_{33} - X_{12} X_{23} X_{31} - X_{13} X_{21} X_{32} + X_{13} X_{22} X_{31}$
(iii) $X_{11} X_{23} X_{32} - X_{11} X_{22} X_{31} - X_{13} X_{21} X_{32} + X_{13} X_{22} X_{31}$

16 cluster variables ↔ almost positive roots of $D_n$.

Each cluster gives rise to a total positivity criteria: a matrix $x \in SL_3$ is TP iff the 4 elements of the given clusters & the 4 coeff variables are all positive at $x$. 
What about the totally non-negative matrices which are not totally positive? Let's take a step back...

There is a decomposition of $G$ into strata (double Bruhat cells) which is "good" w/ respect to total positivity—Zusztig, Fomin & Zelevinsky.

Notation:

Let $G = SL_{r+1}$, $B$ and $B_-$ two "opposite Borel subogs" $B = \begin{pmatrix} * & * & * \\ * & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B_- = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$

$H = B \cap B_- = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$ the "maximal torus",

$W = \text{Norm}_G(H)/H$ the Weyl group, which for $G = SL_{r+1}$ is the symmetric group $S_{r+1}$.

$G$ has two Bruhat decomposition (into double cosets w/ respect to $B$ and $B_-$):

$$G = \bigcup_{u \in W} BuB = \bigcup_{v \in W} B_- v B_-.$$ The double Bruhat cell $G^{u,v} := BuB \cap B_- v B_-$. This is not actually a cell— but it is bijectively isomorphic to a Zariski open subset of an affine space of dimension $r + l(u) + l(v)$—described by saying certain minors must or must not vanish.

We have $G = \bigcup_{u \in W} G^{u,v}$, disjoint union
In what sense is this decomposition good w/ respect to total positivity? If we define $G_{>0}^{u,v} = G_u \cap G_v$ then

**Theorem:** $G_{>0}^{u,v} \subseteq \mathbb{R}_{>0}^{r+l(u)+l(v)}$ (Lusztyk)

Further, $G_{>0}^{u_0, v_0} = G_{>0}$, the set of TP matrices (w/ det 1) are all non-singular and homomorphic to open balls.

This gives a decomposition $G_{>0} = \bigcup_{u,v \in u,v} G_{>0}^{u,v}$ balls.

Lusztyk proved this theorem by giving a parameterization using rational functions that are not necessarily regular on $G_{>0}$. One might hope for something more...

**Def:** A TP-basis for $G_{>0}^{u,v}$ is a collection of regular functions $F = \{f_1, \ldots, f_m\} \subseteq \mathbb{C}[G_{>0}^{u,v}]$ s.t:

(i) $f_1, \ldots, f_m$ are algebraically independent and generate the field of rational functions $\mathbb{C}(G_{>0}^{u,v})$. In particular, $m = r + l(u) + l(v)$

(ii) The map $(f_1, \ldots, f_m) : G_{>0}^{u,v} \rightarrow \mathbb{C}^m$ restricts to a birational isomorphism $U(F) \rightarrow (\mathbb{C}_{>0})^m$ where $U(F) = \{x \in G_{>0}^{u,v} : f_k(x) \neq 0 \text{ for all } k \in \{1, m\}\}$

"$G_{>0}^{u,v}$ looks a lot like $\mathbb{C}^m"$

(iii) The map $(f_1, \ldots, f_m) : G_{>0}^{u,v} \rightarrow \mathbb{C}^m$ restricts to an isomorphism $G_{>0}^{u,v} \rightarrow \mathbb{R}_{>0}^m$.

"$f_1, \ldots, f_m$ provide a total positivity criterion in $G_{>0}^{u,v}$: an element $x \in G_{>0}^{u,v}$ is totally non-negative iff $f_k(x) > 0$ for all $k \in \{1, m\}"
Fomin + Zelevinsky found a large number of total positivity criteria for testing whether a matrix $X \in G^{n \times n}$ is totally nonnegative. Their construction uses a version of the double wiring diagram we saw before, with reduced decompositions for $u$ and $v$, replacing the two reduced decompositions for $w_0$ and $w_0$. 
1. By using this network and its weight matrix, looking at examples, try to guess a combinatorial formula for all of the minors of the weight matrix in terms of path families.

2. Prove that a 3x3 matrix $M$ is totally positive iff the following minors are positive:

\[
\begin{align*}
\Delta_{123,123}(M) \\
\Delta_{123,213}(M) \\
\Delta_{123,312}(M) \\
\Delta_{132,213}(M) \\
\Delta_{132,312}(M) \\
\Delta_{132,123}(M) \\
\Delta_{213,213}(M) \\
\Delta_{213,312}(M) \\
\Delta_{213,123}(M) \\
\Delta_{312,213}(M) \\
\Delta_{312,312}(M) \\
\Delta_{312,123}(M)
\end{align*}
\]

3. For each edge in the graph below, find an algebraic relation that relates the 2 corresponding "clusters."
Notation:

\[ a = x_{11}, \quad b = x_{12}, \quad c = x_{21}, \quad d = x_{22}, \quad e = x_{23}, \quad f = x_{32}, \quad g = x_{33} \]

\[ A = \Delta_{23,23}, \quad B = \Delta_{23,13}, \quad C = \Delta_{13,23}, \quad D = \Delta_{13,13}, \quad E = \Delta_{13,12}, \quad F = \Delta_{12,13}, \quad G = \Delta_{12,12} \]

Minors corresponding to unbounded chambers (appear in all criteria):

\[ x_{13}, x_{31}, \Delta_{12,23}, \Delta_{23,12}, \det(x) \]

FIGURE 8. Total positivity criteria for \( GL_3 \)

two arrangements \( \text{Arr}(i) \) and \( \text{Arr}(\bar{i}) \) whose isotopy types are adjacent in the graph