Invariant subspaces of the Dirichlet shift and harmonically weighted Dirichlet spaces.

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Notation

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \quad f \in \text{Hol}(\mathbb{D}), \; f(z) = \sum_n \hat{f}(n) z^n$

- $H^2 = \{f \in \text{Hol}(\mathbb{D}) : \|f\|_{H^2}^2 = \sum_n |\hat{f}(n)|^2 < \infty\}$

$$\|f\|_{H^2}^2 = \int_{|z|=1} |f(z)|^2 \frac{|dz|}{2\pi}$$

- $D = \{f \in \text{Hol}(\mathbb{D}) : \|f\|_D^2 = \sum_n (n+1)|\hat{f}(n)|^2 < \infty\}$

$$\|f\|_D^2 = \|f\|_{H^2}^2 + \int_{|z|<1} |f'(z)|^2 \frac{dA(z)}{\pi}$$

- $L_a^2 = \{f \in \text{Hol}(\mathbb{D}) : \|f\|_{L_a^2}^2 = \sum_n \frac{|\hat{f}(n)|^2}{n+1} < \infty\}$

$$\|f\|_{L_a^2}^2 = \int_{|z|<1} |f(z)|^2 \frac{dA(z)}{\pi}$$

$$D \subseteq H^2 \subseteq L_a^2$$

Operators

If $\mathcal{H} \in \{H^2, D, L_a^2\}$, then

$(M_z, \mathcal{H})$ is defined by $(M_z f)(z) = z f(z) \; \forall f \in \mathcal{H}$

(M_z, H^2) = unilateral shift

(M_z, D) = Dirichlet shift

(M_z, L_a^2) = Bergman shift

Invariant subspaces

$\mathcal{M} \in \text{Lat}(M_z, \mathcal{H})$ iff $\mathcal{M} \subseteq \mathcal{H}$ is a closed subspace and if $M_z \mathcal{M} \subseteq \mathcal{M}$

$\mathcal{M} \ominus z\mathcal{M} = \mathcal{M} \cap (z\mathcal{M})^{\perp}$ is called the wandering subspace for $\mathcal{M}$

- **Cyclic invariant subspaces:** Let $f \in \mathcal{H}$, $f \neq 0$
  $[f] =$ span $\{f, zf, z^2 f, z^3 f, \dots\} =$
  the cyclic subspace generated by f.
- **Zero-set based invariant subspaces:** Let $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathbb{D}$,
  $\mathcal{M} = I(\{\lambda_n\}) = \{f \in \mathcal{M} : \ f(\lambda_n) = 0 \text{ for all } n\}$.

Then

- $\dim [f] \ominus z[f] = 1$.
- If $I(\{\lambda_n\}) \neq (0)$ is zero-set based, then $\dim \mathcal{M} \ominus z\mathcal{M} = 1$.

Beurling's Theorem, 1948

Theorem

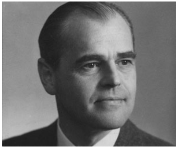
Let $(0) \neq \mathcal{M} \in Lat(M_z, H^2)$, then

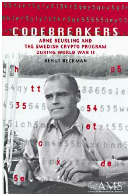
- *$\dim \mathcal{M} \ominus z\mathcal{M} = 1$,*
- *if $\varphi \in \mathcal{M} \ominus z\mathcal{M}$, $\|\varphi\| = 1$, then*

$$\mathcal{M} = [\varphi] = \varphi H^2, \quad \text{so } \frac{\mathcal{M}}{\varphi} = H^2,$$

- *$\varphi \in \mathcal{M} \ominus z\mathcal{M}$, $\|\varphi\| = 1$ is an inner function, i.e. $|\varphi(z)| = 1$ for a.e. $|z| = 1$.*

$$\varphi(z) = cz^n \prod_{k \geqslant 1} \frac{\overline{\lambda}_k}{|\lambda_k|} \frac{\lambda_k - z}{1 - \overline{\lambda}_k z} \quad e^{-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\sigma(t)} \quad (\sigma \text{ singular}, |c| = 1).$$





Arne Beurling (1905-1986)

Bergman space invariant subspaces

Theorem (Apostol, Bercovici, Foias, Pearcy, 1985)

If $n \in \mathbb{N} \cup \{\infty\}$, then there is $\mathcal{M} \in Lat(M_z, L_a^2)$ such that

$$\dim \mathcal{M} \ominus z\mathcal{M} = n.$$

Corollary (Sandwich Theorem, ABFP)

If for all $\mathcal{M}, \mathcal{N} \in Lat(M_z, L_a^2)$, $\mathcal{M} \subseteq \mathcal{N}$, $\dim \mathcal{N} \ominus \mathcal{M} > 1$, there is $\mathcal{K} \in Lat(M_z, L_a^2)$,

$$\mathcal{M} \subsetneq \mathcal{K} \subsetneq \mathcal{N},$$

then every operator on a Hilbert space of dim > 1 has a nontrivial invariant subspace.

Theorem (Hedenmalm, 1991)

If $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathbb{D}$, *if*

$$\mathcal{M} = \{f \in L^2_a : f(\lambda_n) = 0 \text{ for all } n\} \in Lat(M_z, L^2_a),$$

if $\varphi \in \mathcal{M} \ominus z\mathcal{M}$, $\|\varphi\| = 1$, *then*

$$H^2 \subseteq \frac{\mathcal{M}}{\varphi} \subseteq L^2_a.$$

Theorem (Aleman, Richter, Sundberg, 1996)

If $\mathcal{M} \in Lat(M_z, L^2_a)$, *then*

$$\mathcal{M} = [\mathcal{M} \ominus z\mathcal{M}].$$

If $\dim \mathcal{M} \ominus z\mathcal{M} = 1$, *if* $\varphi \in \mathcal{M} \ominus z\mathcal{M}$, $\|\varphi\| = 1$, *then*

$$\mathcal{M} = [\varphi] \text{ and } H^2 \subseteq \frac{\mathcal{M}}{\varphi} \subseteq L^2_a.$$

Dirichlet space invariant subspaces, II

Recall: If $(0) \neq \mathcal{M} \in \text{Lat}(M_z, H^2)$, then $\mathcal{M} = \varphi H^2$, φ inner.
$M_\varphi : H^2 \to \mathcal{M} \subseteq H^2, f \to \varphi f$ is isometric.
Hence $P = M_\varphi M_\varphi^*$ is a projection with kernel $= ker M_\varphi^* = (\text{ran} M_\varphi)^\perp = \mathcal{M}^\perp$, i.e. $P_\mathcal{M} = M_\varphi M_\varphi^*$.

Theorem (McCullough-Trent, 2000)

Let $(0) \neq \mathcal{M} \in Lat(M_z, D)$, *then there are* $\{\varphi_n\} \subseteq M(D)$ *such that*

$$P_\mathcal{M} = \sum_n M_{\varphi_n} M_{\varphi_n}^* \ (SOT)$$

The proof uses that $k_\lambda(z) = \frac{1}{\bar\lambda z} \log \frac{1}{1-\bar\lambda z}$ is a CNP kernel (complete Nevanlinna Pick kernel).

Theorem (Greene, Richter, Sundberg, 2002)

$$ntl\text{-}\lim_{\lambda\to z} \sum_n |\varphi_n(\lambda)|^2 = 1 \text{ for a.e. } z \in \mathbb{T}$$

Dirichlet space invariant subspaces, I

Theorem (Richter-Sundberg 1991-92, Aleman 93)

Let $(0) \neq \mathcal{M} \in Lat(M_z, D)$, *then*

- *dim* $\mathcal{M} \ominus z\mathcal{M} = 1$,
- *if* $\varphi \in \mathcal{M} \ominus z\mathcal{M}$, $\|\varphi\| = 1$, *then*

$$\mathcal{M} = [\varphi] = \varphi D(m_\varphi), \quad \text{and } D \subseteq \frac{\mathcal{M}}{\varphi} = D(m_\varphi) \subseteq H^2,$$

- $\varphi \in \mathcal{M} \ominus z\mathcal{M}$, $\|\varphi\| = 1$ *is a contractive multiplier, i.e.* $\|\varphi f\| \leqslant \|f\|\ \forall f \in D$, *in particular* $|\varphi(z)| \leqslant 1$ *for* $|z| < 1$.

$$D(\mu) = \{f \in \text{Hol}(\mathbb{D}) : \int_{|z|<1} |f'(z)|^2 \int_{|\zeta|=1} \frac{1-|z|^2}{|z-\zeta|^2} d\mu(\zeta) \frac{dA(z)}{\pi} < \infty\}$$

$$dm_\varphi(z) = |\varphi(z)|^2 \frac{|dz|}{2\pi}$$

Theorem (Shimorin, 2002)

The reproducing kernel for each harmonically weighted Dirichlet space $D(\mu)$ *is a CNP kernel.*

Careful: It is not true, that if $\mathcal{H}$ has a CNP kernel and if $\mathcal{M} \in \text{Lat}(M_z, \mathcal{H})$, then $\mathcal{M}$ has a CNP kernel.

Corollary

Let $\mathcal{M}, \mathcal{N} \in Lat(M_z, D(\mu))$, *with*

$$(0) \neq \mathcal{M} \subseteq \mathcal{N} \subseteq D(\mu)$$

and extremal functions $\varphi_\mathcal{M}$, $\varphi_\mathcal{N}$, *then*

$$D(\mu) \subseteq \frac{\mathcal{N}}{\varphi_\mathcal{N}} = D(\mu_{\varphi_\mathcal{N}}) \subseteq D(\mu_{\varphi_\mathcal{M}}) = \frac{\mathcal{M}}{\varphi_\mathcal{M}} \subseteq H^2.$$

Two-isometric operators

Definition

We say an operator $T \in \mathcal{B}(\mathcal{H})$ is analytic, if $\bigcap_n T^n\mathcal{H} = (0)$.

If $\mathcal{H} \subseteq \mathrm{Hol}(\mathbb{D})$, then $(M_z, \mathcal{H})$ is analytic.

Corollary

Let $T \in \mathcal{B}(\mathcal{H})$ be isometric and analytic, then $T = S$ is a unilateral shift of multiplicity $\dim \mathcal{H} \ominus T\mathcal{H}$

Corollary

Let $T = (M_z, H^2)$, thus T is isometric and analytic, then $\forall \mathcal{M} \in LatT$, $\mathcal{M} \neq (0)$ we have

$$T|\mathcal{M} \quad \text{is isometric and analytic,}$$

hence $T|\mathcal{M}$ is unitarily equivalent to a unilateral shift of multiplicity $\dim \mathcal{M} \ominus T\mathcal{M}$.

Thus, Beurling's theorem follows essentially by showing that $\dim \mathcal{M} \ominus T\mathcal{M} = 1$.

Wold decomposition

Theorem

Let $T \in \mathcal{B}(\mathcal{H})$ be isometric, i.e. $\|Tx\| = \|x\|\ \forall x \in \mathcal{H}$ (equivalently, $\langle Tx, Ty\rangle = \langle x, y\rangle\ \forall x, y \in \mathcal{H}$). Then

$$T = S \oplus U \ \text{ with respect to } \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2,$$

U unitary (=isometric and onto), $\mathcal{H}_2 = \bigcap_n T^n\mathcal{H}$
S unilateral shift of multiplicity $\dim \mathcal{H} \ominus T\mathcal{H}$

$T\mathcal{H} = S\mathcal{H}_1 \oplus U\mathcal{H}_2 = S\mathcal{H}_1 \oplus \mathcal{H}_2$

$$\bigcap_n T^n\mathcal{H} = \bigcap_n S^n\mathcal{H}_1 \oplus \mathcal{H}_2 = (0) \oplus \mathcal{H}_2$$

If $\mathcal{K} = \mathcal{H} \ominus T\mathcal{H} = \mathcal{H}_1 \ominus S\mathcal{H}_1$, then

$$\mathcal{H}_1 = \mathcal{K} \oplus S\mathcal{K} \oplus S^2\mathcal{K} \oplus \ldots$$

Thus the name wandering subspace (Halmos).

(M_z, D) is a 2-isometry

$f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n,\ \ zf(z) = \sum_{n=1}^{\infty} \hat{f}(n-1)z^n,$

$\|f\|_D^2 = \sum_{n=0}^{\infty}(n+1)|\hat{f}(n)|^2$

$\|zf\|_D^2 = \sum_{n=1}^{\infty}(n+1)|\hat{f}(n-1)|^2 = \sum_{n=0}^{\infty}(n+2)|\hat{f}(n)|^2$

$\|zf\|_D^2 - \|f\|_D^2 = \sum_{n=0}^{\infty}|\hat{f}(n)|^2 = \|f\|_{H^2}^2$

$$\|z^2f\|_D^2 - \|zf\|_D^2 = \|zf\|_{H^2}^2 = \|f\|_{H^2}^2 = \|zf\|_D^2 - \|f\|_D^2$$

Definition (Agler)

$T \in \mathcal{B}(\mathcal{H})$ is a two-isometry, if and only if

$$\|T^2x\|^2 - \|Tx\|^2 = \|Tx\|^2 - \|x\|^2 \ \ \forall x \in \mathcal{H}$$

Theorem (Wold decomposition for 2-isos)

Let $T \in \mathcal{B}(\mathcal{H})$ be a 2-isometry.
Then

$$T = S \oplus U \text{ with respect to } \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2,$$

U unitary, $\mathcal{H}_2 = \bigcap_n T^n\mathcal{H}$
S analytic 2-isometry

Proof.

Lemma (proof later)

$\|Tx\| \geqslant \|x\| \ \ \forall x \in \mathcal{H}$

Verify that $T\mathcal{H}_2 = \mathcal{H}_2$, then $T|\mathcal{H}_2$ is an invertible 2-isometry,and $(T|\mathcal{H}_2)^{-1}$ is a 2-isometry.
Then by the Lemma $T|\mathcal{H}_2 = U$ unitary.
Finally show that $\mathcal{H}_2$ is reducing using U unitary, T 2-isometry. □

Theorem (Wandering subspace theorem)

If S is an analytic 2-isometry, and if

$$\mathcal{K} = \mathcal{H} \ominus S\mathcal{H} = (\text{ran } S)^{\perp} = \text{ ker } S^*,$$

then

$$\mathcal{H} = [\mathcal{K}]_S = \bigvee_{n=0}^{\infty} S^n\mathcal{K}.$$

In particular, if $\mathcal{M} \in \text{Lat}\,T$ with

$$\dim \mathcal{M} \ominus T\mathcal{M} = 1$$

then for $\varphi \in \mathcal{M} \ominus T\mathcal{M}$, $\|\varphi\| = 1$ we have

$$\mathcal{M} = [\varphi].$$

Theorem

Let $T \in \mathcal{B}(\mathcal{H})$, then the following are equivalent:

- *T is an analytic 2-isometry with dim ker $T^* = 1$,*
- *T is unitarily equivalent to $(M_z, D(\mu))$ for some $\mu \in M_+(\mathbb{T})$.*

$\|f\|^2_{D(\mu)} = \|f\|^2_{H^2} + \int_{|\zeta|=1} D_\zeta(f)d\mu(\zeta)$

$$D_\zeta(f) = \int_{|z|=1} \frac{|f(z) - f(\zeta)|^2}{|z-\zeta|^2}\frac{|dz|}{2\pi} = \int_{|z|<1} |f'(z)|^2 \frac{1-|z|^2}{|z-\zeta|^2}\frac{dA(z)}{\pi}$$

If $(0) \neq \mathcal{M} \in \text{Lat}(M_z, D(\mu))$, if dim $\mathcal{M} \ominus z\mathcal{M} = 1$, then

$$M_z|\mathcal{M} \text{ is u. e. to } (M_z, D(\sigma)).$$

We will see that $\mathcal{M} = \varphi D(\mu_\varphi)$.

Lemma

If T is a 2-isometry, then $\|Tx\| \geqslant \|x\|$ for all $x \in \mathcal{H}$

Proof.

$\|T^2x\|^2 - \|Tx\|^2 = \|Tx\|^2 - \|x\|^2$
$\|T^kx\|^2 - \|T^{k-1}x\|^2 = \|Tx\|^2 - \|x\|^2$

$$\begin{aligned}\|T^nx\|^2 - \|x\|^2 &= \sum_{k=1}^{n} \|T^kx\|^2 - \|T^{k-1}x\|^2 \\ &= \sum_{k=1}^{n} \|Tx\|^2 - \|x\|^2 \\ &= n(\|Tx\|^2 - \|x\|^2)\end{aligned}$$

$\|Tx\|^2 - \|x\|^2 \geqslant -\frac{1}{n}\|x\|^2 \to 0$ as $n \to \infty$ □

Thus if T is a 2-isometry, then

$$T^*T - I \geqslant 0,$$

so we define

$$D = (T^*T - I)^{1/2}$$

defect operator

We have $\|Dx\|^2 = \langle D^2x, x\rangle = \|Tx\|^2 - \|x\|^2$
and
$\|DTx\| = \|Dx\|$ and $\|DT^kx\| = \|Dx\|$

hence "T is isometric with respect to $\|x\|_* = \|Dx\|$"

Theorem

If T is a 2-iso with defect operator D, if $x_0 \in \mathcal{H}$, then there exists $\mu \in M_+(\mathbb{T})$ such that

$$\|Dq(T)x_0\|^2 = \int |q|^2 d\mu \quad \forall q \text{ poly}.$$

Proof.

For $n \geqslant 0$ define

$$M_n = \langle DT^n x_0, Dx_0\rangle$$

and for $n < 0$ set

$$M_n = \langle Dx_0, DT^{|n|}x_0\rangle$$

Then $M_{-n} = \overline{M_n}$ for all n.
Claim: $\{M_n\}$ is a moment sequence, i.e.
$\exists \mu \in M_+(\mathbb{T})$ such that $M_n = \int z^n d\mu$ for all n

□

If $M_n = \int z^n d\mu$ for all n, then for any polynomial $q(z) = \sum_n \hat{q}(n) z^n$ we have

$$\begin{aligned}
\int |q|^2 d\mu &= \sum_{n,m} \hat{q}(n)\overline{\hat{q}(m)} \int z^{n-m} d\mu \\
&= \sum_{n,m} \hat{q}(n)\overline{\hat{q}(m)} M_{n-m} \\
&= \sum_{n\geqslant 0}\sum_{m=0}^{n} \hat{q}(n)\overline{\hat{q}(m)} \langle DT^{n-m}x_0, Dx_0\rangle \\
&\quad + \sum_{n\geqslant 0}\sum_{m>n} \hat{q}(n)\overline{\hat{q}(m)} \langle Dx_0, DT^{m-n}x_0\rangle \\
&= \sum_{n\geqslant 0}\sum_{m=0}^{n} \hat{q}(n)\overline{\hat{q}(m)} \langle DT^{n}x_0, DT^m x_0\rangle \\
&\quad + \sum_{n\geqslant 0}\sum_{m>n} \hat{q}(n)\overline{\hat{q}(m)} \langle DT^n x_0, DT^{m}x_0\rangle \\
&= \|Dq(T)x_0\|^2
\end{aligned}$$

Repeating:

If $M_n = \int z^n d\mu$ for all n, then for any polynomial $q(z) = \sum_n \hat{q}(n) z^n$ we have

$$\int |q|^2 d\mu = \sum_{n,m} \hat{q}(n)\overline{\hat{q}(m)} M_{n-m} = \|Dq(T)x_0\|^2$$

The equality of the RHS with the middle term also shows that $\{M_n\}$ is a moment sequence by the following well-known theorem.

Theorem (Moment sequences)

Let $\{M_n\}_{n\in\mathbb{Z}} \subseteq \mathbb{C}$.
The following are equivalent:

- $\exists\, \mu \in M_+(\mathbb{T})$ *with* $M_n = \int z^n d\mu$,
- $\{M_n\}_{n\in\mathbb{Z}}$ *is positive definite, i.e.*
  $\forall\, N \in \mathbb{N} \forall a_1, ..., a_N \in \mathbb{C}$ *we have* $\sum_{n,m} a_n \overline{a}_m M_{n-m} \geqslant 0$.

Proof.

We assume the second condition and need to show the existence of the measure μ.
Define a linear functional on the trigonometric polynomials by $L(z^n) = M_n$. We will show that L extends to be a positive linear functional on $C(\mathbb{T})$, then the result will follow from the Riesz representation theorem.
Fact (Fejer-Riesz theorem): If $p(e^{it}) \geqslant 0$ is a trig poly, then there is an analytic poly q with $p = |q|^2$.
Thus $L(p) = L(|q|^2) \geqslant 0$ by hypothesis for any nonnegative trig poly p.
Now use that the trig polys are dense in $C(\mathbb{T})$.

□