Spring 2020

EMISSARY

Mathematical Sciences Research Institute

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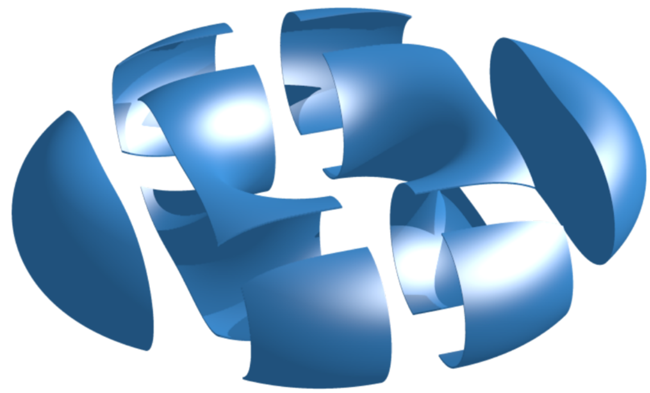
Higher Categories and Categorification

David Gepner and Claudia Scheimbauer

Categories arise everywhere in mathematics: things we study usually come with a natural notion of maps between them, such as sets and functions between them, vector spaces and linear transformation, groups and homomorphisms, … In practice, there are often many natural choices of maps: for instance, in geometry, one can take for morphisms all functions, continuous functions, differentiable functions, smooth functions, polynomial functions, and others.

The first concepts in category theory were formulated by Eilenberg and Mac Lane in the early 1940s in an attempt to make precise the notion of *naturality* in mathematics. They were driven by examples coming from algebraic topology: in particular, studying how points, paths, triangles, and so on, relate when we continuously deform the spaces in which they live. (See the figure on page 4.)

Formally, a category $\mathcal{C}$ consists of classes of objects $\{A, B, C, \ldots\}$ and morphisms $\{f, g, h, \ldots\}$ between objects, together with a rule for composing (uniquely) morphisms with compatible source and target: if $f : A \to B$ and $g : B \to C$ are morphisms then $g \circ f : A \to C$ is again a morphism. This composition rule is associative — that is, $(h \circ g) \circ f = h \circ (g \circ f)$ — and has identities.



Cutting manifolds into simple pieces is one way higher categories arise naturally.

Categories arise from geometric objects themselves. From a surface, or more generally a topological space X, we can produce a

(continued on page 4)

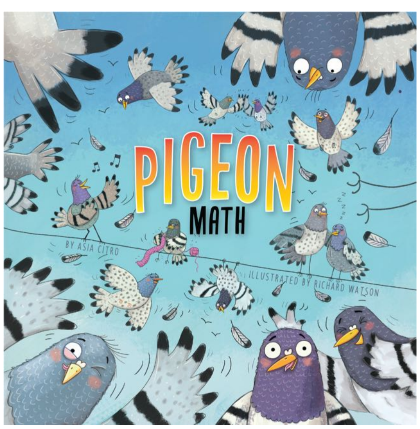
Mathical Inspires Children Nationwide

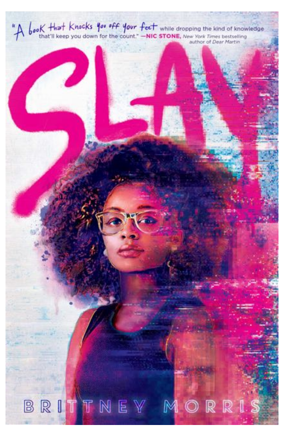
The **Mathical Book Prize**, now in its sixth year, is growing in scope and reach through partnerships and recognition across the country. Mathical books can be poetry, biography, fiction, nonfiction — anything both "lyrical and logical."

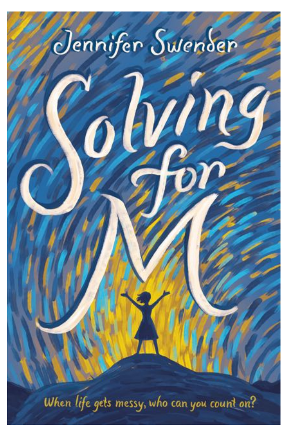
They are math-inspiring literary books of all sorts and types that appeal to children's wide-ranging interests: from magic to baseball to cooking to nature to mystery to adventure, and much more. They are engaging and energizing — and we hope you investigate them if you haven't already!

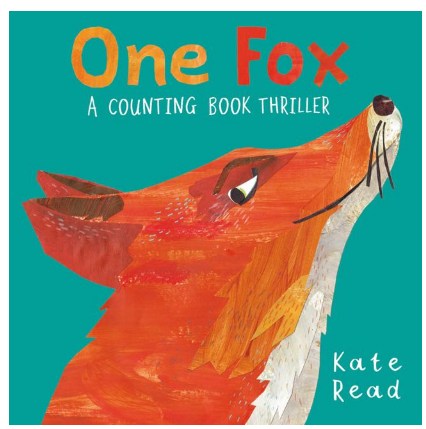
You can find a list of this year's Mathical winnners (pictured below) along with news about Mathical's reach to low income schools and students as well as its growing network of partners on page 9.

In response to the COVID-19 pandemic, **MSRI programs and workshops for Spring 2020 have moved online**. For news about changes to the institute's Summer 2020 activities, visit msri.org. We are very grateful to all the organizers and speakers for their support, patience, and invaluable contributions to this transition.









The View from MSRI

David Eisenbud, Director

Over the last year and a half, I've been reporting in this column on the state of our National Science Foundation recompetition proposal. I'm happy now to announce what I think is the penultimate chapter: MSRI is being recommended by the Division of Mathematical Sciences for a new grant of $5 million per year for the next five years, the maximum allowed in the call for proposals!

Needless to say, I'm both delighted and relieved at the security this gives us, temporary though it is. The NSF grant is now about one-half of our budget — and about two-thirds of our direct science budget — so its loss would have been catastrophic. I believe that the only long-term path to security is developing a larger endowment. We've made a start — the endowment we've built up over the last dozen years now provides about 10% of our total support — but we have a long way to go until it can guarantee the long-term ability of MSRI to continue serving the mathematical sciences community.

Scientific Activities

Our programs this spring are "Higher Categories and Categorification" and "Quantum Symmetries," the latter having much to do with monoidal categories. As a student of Saunders Mac Lane, I had a fair grounding in category theory. I didn't look beyond 2-categories at the time, old-fashioned in this day of infinity-categories. I wonder whether Mac Lane would have been surprised that the push for such an extension would come from homotopy theory. Some things don't change so much, though: I enjoy seeing lecturers draw the pentagonal coherence axiom for monoidal categories that Mac Lane discovered and taught me 50+ years ago.

A scientific activity of a very different kind centers around the annual prize for mathematical contributions to finance and economics that we give jointly with the CME Group (formerly called the Chicago Mercantile Exchange). The winner this year is a star of the economics and machine learning communities, Susan Athey (see page 3). The award ceremony and panel, originally planned for April 7, will be rescheduled, with more information expected in Summer 2020.

Underrepresented Groups in Mathematics

MSRI has substantially expanded its effort to help women and other currently underrepresented groups in mathematics stay engaged in research. One such initiative is simply called Summer Research in Mathematics (SRIM). Small groups consisting predominantly of women mathematicians that have an established research collaboration apply to come for a couple of weeks to continue their work. We provide comprehensive support for childcare, which has turned out to be a tremendous draw: 80 groups, comprised of nearly 300 researchers, applied this year, almost double the numbers from last year. Thanks to generous gifts from the Lyda Hill and McGovern Foundations, we have been able to accept about 25% of those applications.

Another such initiative is the African Diaspora Joint Mathematics Workshop (ADJOINT). It is a two-week summer program for researchers with Ph.D.s in the mathematical sciences interested in conducting research in a collegial environment. We ran a pilot program with NSF support in the summer of 2019, and this summer, depending on COVID-19, we will host or e-host over 20 African-American mathematicians through generous support by the Sloan Foundation and the NSA. The participants will work in small groups on five different projects, each with a well-established research leader. See msri.org/adjoint for more information.

SRIM and ADJOINT, together with four two-week summer graduate schools and the six-week REU program MSRI-UP, now keep MSRI buzzing during the summer months more than ever before!

Popularization and Public Understanding

Readers of this column have heard before that we were working on the film *Secrets of the Surface: The Mathematical Vision of Maryam Mirzakhani*. It was directed by George Csicsery (who also directed *N is a Number* about Paul Erdös, *Counting from Infinity* about Yitang Zhang, and *Navajo Math Circles*). The film is done! It had its world premiere at the Joint Mathematics Meetings this year in Denver before a standing-room-only crowd, and it will soon have screenings all over the world. See the article on the back page for more information.

Another of MSRI's ventures in popularization, the Mathical Book Prize, has achieved a new level of success this year, through a continuing grant from the Firedoll Foundation and a new grant from the McGovern Foundation. For example, the centennial meeting of the National Council of Teachers of Mathematics, April 1–4 in Chicago, was poised to feature a large group of prizewinning Mathical authors reading and talking about their work. (We hope that Mathical will continue to be among the rescheduled components of the meeting.) MSRI is now able to enlarge the distribution of the books, for example with grants to school libraries in underserved settings. There's an in-depth look at Mathical's partners and its reach to underserved communities, along with a list of this year's winners, on page 9.

There's lots more going on, both at and through MSRI. You can keep track of us on the web or, better, by coming to participate in a workshop or program. I hope to see you here before too long!

Contents

CME Group–MSRI Prize

The 14th annual CME Group–MSRI Prize in Innovative Quantitative Applications was awarded to **Susan Athey** (pictured). The award ceremony and panel, originally scheduled for April 7, will be rescheduled, with more information expected in Summer 2020.

Dr. Athey is the Economics of Technology Professor; Professor of Economics (by courtesy), School of Humanities and Sciences; and Senior Fellow, Stanford Institute for Economic Policy Research, Stanford Business School. Her current research focuses on the economics of digitization, marketplace design, and the intersection of econometrics and machine learning. She has worked on several application areas, including timber auctions, internet search, online advertising, the news media, and the application of digital technology to social impact applications. As one of the first "tech economists," she served as consulting chief economist for Microsoft Corporation for six years, and now serves on the boards of Expedia, Lending Club, Rover, Turo, and Ripple, as well as nonprofit Innovations for Poverty Action. She is the founding director of the Golub Capital Social Impact Lab at Stanford GSB, and associate director of the Stanford Institute for Human-Centered Artificial Intelligence.

The annual CME Group–MSRI Prize is awarded to an individual or a group to recognize originality and innovation in the use of mathematical, statistical, or computational methods for the study of the behavior of markets, and more broadly of economics. You can read more about the CME Group–MSRI Prize at at tinyurl.com/cme-msri.

Congressional Briefings

MSRI and the American Mathematical Society (AMS) host two congressional briefings on mathematical topics each year in Washington, DC, to inform members of Congress and Congressional staff about new developments made possible through federal support of basic science research. On December 5, 2019, **Jill Pipher** (pictured), Vice President for Research and Elisha Benjamin Andrews Professor of Mathematics at Brown University, spoke to congressional staff and the public about her research in her talk "No Longer Secure: Cryptography in the Quantum Era." Pipher traced the impact of modern cryptography on network security, cybersecurity, financial transactions, private communications, and digital currencies, outlining the complex environment the U.S. faces at the beginning of the quantum era. Learn more and view upcoming events: msri.org/congress.

Nick Dentamaro / Brown University

Named Positions — Spring 2020

MSRI is grateful for the generous support that comes from endowments and annual gifts that support faculty and postdoc members of its programs each semester.

Chern, Eisenbud, and Simons Professors

Quantum Symmetries
Terry Gannon, University of Alberta
Vaughan Jones, Vanderbilt University
Victor Ostrik, University of Oregon
Sorin Popa, CNRS / University of California, Los Angeles
Kevin Walker, Microsoft Research Station Q
Sarah Witherspoon, Texas A&M University

Higher Categories and Categorification
Clark Barwick, University of Edinburgh
David Ben-Zvi, University of Texas, Austin
David Gepner, University of Melbourne
Teena Gerhardt, Michigan State University
Emily Riehl, Johns Hopkins University
Ulrike Tillmann, Oxford University

Named Postdoctoral Fellows

Quantum Symmetries
Della Pietra: Colleen Delaney, Indiana University
Huneke: Cris Negron, University of North Carolina
Strauch: David Reutter, Max-Planck-Institut für Mathematik

Higher Categories and Categorification
Berlekamp: Nicolle Sandoval González, UCLA
Strauch: David Reutter, Max-Planck-Institut für Mathematik
Viterbi: Alexander Campbell, Macquarie University

Clay Senior Scholars for 2020–21

The Clay Mathematics Institute (www.claymath.org) Senior Scholar awards provide support for established mathematicians to play a leading role in a topical program at an institute or university away from their home institution. Here are the Clay Senior Scholars who will work at MSRI in 2020–21.

Random and Arithmetic Structures in Topology (Fall 2020)
Uri Bader, Weizmann Institute

Decidability, Definability and Computability in Number Theory (Fall 2020)
François Loeser, Université Pierre et Marie Curie

Mathematical Problems in Fluid Dynamics (Spring 2021)
Jean-Marc Delort, Université Paris 13

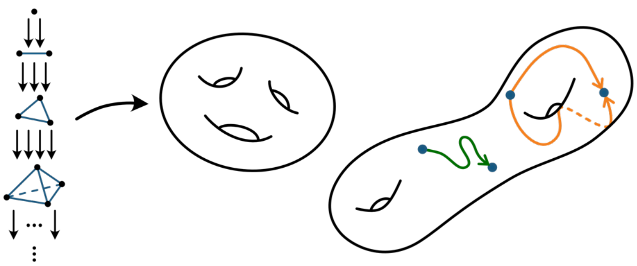
Pacific Journal of Mathematics

*Founded in 1951, **The Pacific Journal of Mathematics** has published mathematics research for more than 60 years. PJM is run by mathematicians from the Pacific Rim and aims to publish high-quality articles in all branches of mathematics, at low cost to libraries and individuals. PJM publishes 12 issues per year. Please consider submitting articles to PJM. The process is easy and responses are timely. See* msp.org/publications/journals/#pjm.

Higher Categories and Categorification

(continued from page 1)

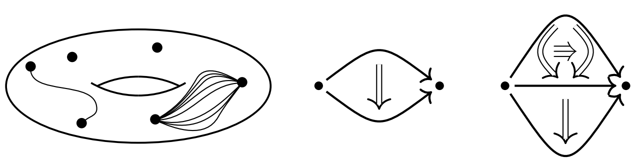
category whose objects are the points of X, and whose morphisms (between a pair of points) are the paths in X (formally, continuous maps $[0,1] \to X$) between these points. Composition is by concatenation of paths. Observe that there is a subtlety here: namely, the composition of the path $\alpha : [0,1] \to X$ followed by the path $\beta : [0,1] \cong [1,2] \to X$ is a map $\gamma : [0,2] \to X$, which we may view as a path after choosing a reparametrization $[0,2] \cong [0,1]$. The reader may like to check for themselves that this composition is *not* associative.



Surfaces produce a "category" whose objects are the points of the surface and morphisms are the paths on the surface.

From Categories to Higher Categories

Historically, the first attempt to fix this was to redefine the morphisms as equivalence classes of paths (path homotopies), leading to the fundamental groupoid $\Pi_1(X)$. However, it is a fundamental principle in higher category theory, the idea of which goes back at least to Grothendieck, to retain this information, leading to the fundamental ∞-groupoid $\Pi_\infty(X)$ (which retains the homotopy type of X). In particular, we now view a path homotopy as a morphism between morphisms, which we call a 2-morphism. More generally, an n-morphism is a morphism between $(n-1)$-morphisms. A *higher category* has objects, and n-morphisms for all n, which compose appropriately.



(a) (b) (c)

(a) Two objects, a morphism, and a 2-morphism in the fundamental ∞-groupoid of the torus; (b) a 2-morphism in a higher category; (c) a composition of a 3-morphism and a 2-morphism in a higher category.

Higher categories also arise when keeping track of the symmetries of an object. For example, let us look at commutativity of a binary operation, say, addition or multiplication of any two real numbers a and b. If a and b are natural numbers, we can look at two sets A and B which have cardinality a and b, respectively. The identity $a \cdot b = b \cdot a$ tells us that the cardinalities of $A \times B$ and $B \times A$ are equal. But we can say more. If $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$, then

$$A \times B = \{(a_1, b_1), (a_2, b_1), \ldots\}$$
$$B \times A = \{(b_1, a_1), (b_2, a_1), \ldots\},$$

so we have a bijection $\varphi_{A,B} : A \times B \to B \times A$, which is not the identity. These bijections are natural in A and B, which we now explain.

Let $\mathcal{Fin}$ denote the category of finite sets and bijections. The assignments $(A,B) \mapsto A \times B$ and $(A,B) \mapsto B \times A$ are also compatible with morphisms, which is summarized by saying they are *functors* $\mathcal{Fin} \times \mathcal{Fin} \to \mathcal{Fin}$. The assignments $(A,B) \mapsto \varphi_{A,B}$ form a natural transformation of functors, which means that any maps $f : A \to A'$ and $g : B \to B'$ induce maps between both products, and all of these maps are compatible; that is, the square

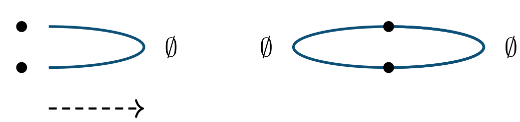
$$\begin{tikzcd}
A \times B \arrow[r, "\varphi_{A,B}"] \arrow[d, "f \times g"] & B \times A \arrow[d, "g \times f"] \\
A' \times B' \arrow[r, "\varphi_{A',B'}"] & B' \times A'
\end{tikzcd}$$

commutes (going around either way yields the same result). This illustrates that categories themselves form a higher category: objects are categories, morphisms are functors, and 2-morphisms are natural transformations.

Geometry and Field Theory

Cobordism categories. Any 1-dimensional smooth manifold with boundary is a disjoint union of copies of the circle and the closed interval. We can view them as morphisms in a category in the following way: choose a decomposition of its boundary into two parts, which we will call incoming and outgoing boundaries, respectively, to obtain a *cobordism*. The objects in the category are finite sets of points (viewed as a 0-dimensional manifold); in particular, the in- and outgoing boundaries are objects. We compose by gluing along a common boundary. (Technically, we should identify diffeomorphic cobordisms for this to be well-defined.)

For example, consider the following two diagrams:



The left figure depicts a morphism from two points to the empty set, where the arrow indicates the direction. The right figure depicts a composition of two elbows along two points, giving a circle. The viewpoint of thinking of this as cutting a circle into easier pieces will turn out to be useful below.

This category is called the category of cobordisms. It comes in many different flavors: we can increase the dimension n of the manifolds and add orientations, framings, spin structures, or even non-topological, that is, geometric, decorations. We will use the notation $\mathcal{Bord}_n$ in the following to denote generically one of these cobordism categories or its higher cousins, which we will encounter below.

Furthermore, taking disjoint unions gives a *symmetric monoidal structure*.

Duality. The symmetric monoidal (higher) category $\mathcal{B}ord_n$ enjoys a universal property, relating the geometry to algebraic properties. To see this, let us begin with the prototypical example.

Consider the category of (real) vector spaces $\mathcal{V}ect$. Let's look at the object $\mathbb{R}^n$, and observe that we have linear maps

$$A\colon \mathbb{R}^{n^2} \cong \mathbb{R}^n \otimes \mathbb{R}^n \cong \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}) \otimes \mathbb{R}^n \to \mathbb{R}, \quad (\varphi, v) \mapsto \varphi(v),$$
$$B\colon \mathbb{R} \to \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^n) \cong \mathbb{R}^n \otimes \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}), \quad 1 \mapsto \mathrm{id} \in \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^n).$$

Let us pick the standard basis $\{e_i\}$ of $\mathbb{R}^n$ and denote vectors in the dual basis by e^i, so that $e^i(e_j)$ is 1 if $i = j$ and 0 otherwise. They form a basis of $\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}) \cong \mathbb{R}^n$. The maps A and B enjoy a nice property (which we will label †):

$$\begin{array}{ccccc} \mathbb{R}^n & \xrightarrow{B\otimes \mathrm{id}} & \mathbb{R}^n \otimes \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}) \otimes \mathbb{R}^n & \xrightarrow{\mathrm{id}\otimes A} & \mathbb{R}^n, \\ e_j & \longmapsto & (\sum_i e_i \otimes e^i) \otimes e_j & \longmapsto & \sum_i e_i \otimes (e^i(e_j)) = e_j, \end{array}$$

so this composition is the identity! Similarly, we could have started with $\mathrm{Hom}(\mathbb{R}^n,\mathbb{R})$ and first used $\mathrm{id}\otimes B$ and then $A \otimes \mathrm{id}$, and we would also have gotten the identity. This property exhibits $\mathrm{Hom}(\mathbb{R}^n,\mathbb{R})$ as a *dual* of $\mathbb{R}^n$.

We can now ask the question: when does a vector space V have a dual in the sense that there is another vector space W and maps $\mathrm{coev}: \mathbb{R} \to V \otimes W$ and $\mathrm{ev}: W \otimes V \to \mathbb{R}$, satisfying the triangle identities (namely, the two compositions alluded to above are the identity). The answer is: exactly for the finite-dimensional vector spaces!

Our definition of having a dual used that the category of vector spaces has a tensor product $\otimes$, so we can repeat the definition in any monoidal category $\mathcal{C}$, a monoidal ∞-category, or even in a monoidal (∞, n)-category. Moreover, it is an important insight that this notion is an instance of having adjoints in a higher category (which is analogous to a functor having an adjoint).

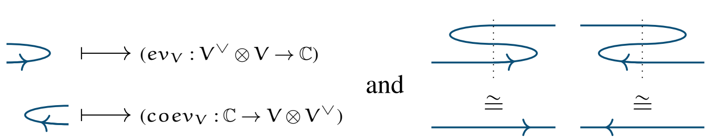
This is the lowest-dimensional example of a beautiful and deep connection between higher category theory and smooth manifolds, called the cobordism hypothesis.

Topological field theories. A mathematical framework for field theories following Atiyah and Segal is to define it to be a symmetric monoidal functor

$$Z: \mathcal{B}ord_n \longrightarrow \mathcal{V}ect,$$

which, roughly speaking, assigns a vector space of states to an object (space) and a linear evolution operator to a cobordism (spacetime).

The cobordism hypothesis now says that a one-dimensional oriented topological field theory corresponds exactly to a vector space V which has a dual. Concretely, it assigns



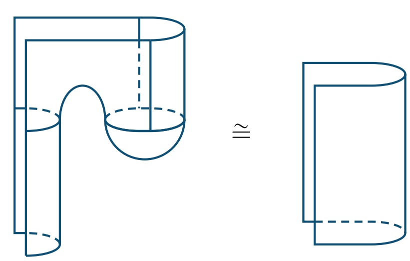
unravel to exactly our property (†) from above.

There are two ways in which higher categories appear naturally when studying bordism categories. First, when increasing the dimension, we might like to cut our manifolds into simple pieces by cutting in several directions, as shown in the figure of the torus on the cover. This naturally has the structure of an n-*category*, for which we have objects, morphisms, and morphisms between morphisms (called 2-morphisms), and so on. Then each of the pieces in the figure are 2-morphisms in this higher category of cobordisms. The boundary of each of these pieces is one-dimensional, albeit in an interesting way. One can read off an incoming and outgoing part on the left and right, and each of these parts themselves look like the pictures we had before.

Second, we might want to consider a moduli space of n-dimensional manifolds rather than a set thereof, by including information about diffeomorphisms in the form of a classifying space. The motivation from physics for this is that the theory should depend smoothly on the space or spacetime. This is implemented by using (∞, n)-categories as the categorical framework. We denote the symmetric monoidal (∞, n)-category of cobordisms equipped with n-framings by $\mathcal{B}ord_n^{fr}$. A fully extended n-dimensional topological field theory valued in $\mathcal{C}$ is a symmetric monoidal functor from $\mathcal{B}ord_n^{fr}$ to $\mathcal{C}$. With this at hand, the Baez–Dolan–Lurie cobordism hypothesis reads as follows:

> *Cobordism hypothesis.* Evaluation at a point with a standard n-framing gives an equivalence of ∞-groupoids between fully extended n-dimensional topological field theories and n-dualizable objects in $\mathcal{C}$.

The condition of being n-dualizable is a generalization of having a dual, and should be thought of as a categorical, hence algebraic condition. In $\mathcal{B}ord_n$ the 2-dualizability of a point can be visualized by the following diffeomorphism between (compositions of) cobordisms:



Thus, a higher categorical property can be understood via geometry, and vice versa.

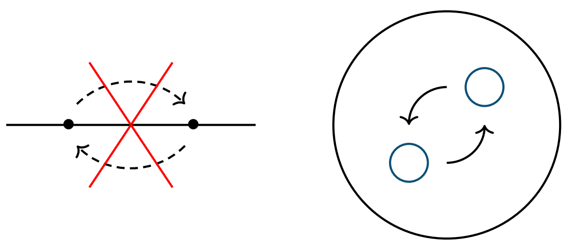
Several research members of this semester's program work on projects related to the cobordism hypothesis; on the proof and variations, on verifying dualizability in various settings and applications, in particular to topological field theories, knot theory, and representation theory. The latter also has interesting connections to research interests of our partner program on Quantum Symmetries.

Algebra: Groups and Spheres

If X is a space with a chosen basepoint $x_0 \in X$, elements of the k^{th} homotopy group $\pi_k(X, x_0)$ are represented by continuous maps $f\colon D^k \to X$, which send the boundary to x_0. (Any two such maps represent the same elements if they are homotopic, that is, if one can be continuously deformed into the other.) In fact, the elements of $\pi_k(X, x_0)$ form a group, which is even commutative if $k > 1$. The conceptual reason for this is fundamentally higher categorical in nature, as we illustrate below.

The group structure is given as follows: consider a pair of elements of $\pi_k(X, x_0)$, represented by a pair of continuous maps $f, g\colon D^k \to X$ that send everything on the boundary of these disks to x_0. Now embed these two disks disjointly into a bigger disk. The product of these two elements is represented by the continuous map $D^k \to X$ obtained by sending everything outside the little disks to the basepoint x_0 of X. However, this depends on the embedding of the little disks into the bigger disk. By deforming the embedding, we can assume that one of the disks sits at the origin. The other disk is free to move around the complement, which is homotopic to S^{k-1}.

If $k = 1$, this is a discrete space, so that we cannot move the free disk past the fixed one, and indeed the so-called fundamental group $\pi_1(X, x_0)$ need not be commutative! For $k > 1$, this is a connected space, so we can always move the free disk past the fixed one, but in an S^{k-1}-parameter space worth of ways. This space is invisible in $\pi_k(X, x_0)$ because we identified homotopic maps, making it commutative. (This is the classical Eckmann–Hilton argument.) However, if we retain this information, which is natural from the perspective of higher category theory, we obtain something that is more than associative, but not quite commutative.



Moving disks in the discrete space S^0 (left) and in the connected space S^{k-1} (right).

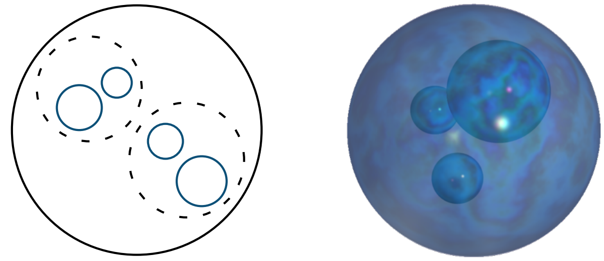
As $k \to \infty$, the space S^{k-1} becomes increasingly connective, and in the limit, S^∞ is contractible, so that (in the appropriate higher categorical sense) there is a unique witness for the commutativity relation $[\alpha] \cdot [\beta] = [\beta] \cdot [\alpha]$. This behavior can be formalized in terms of the *little* k*-disks operad*, a fundamental object of study by various members of this semester's program. It records the embeddings of k-dimensional disks, allowing for successive embeddings, as in the figure at the top of the next column.

Higher Algebraic Structures

Algebraic structures, such as groups and rings, also exist higher categorically. The higher categorical notion of a group is a (based) loop space. That is, the space

$$\Omega X := \mathrm{Map}_*(S^1, X)$$

of loops in X, based at the basepoint $x_0 \in X$, admits a group structure in which the multiplication is given by concatenation of loops. This is only a group in the higher categorical sense, as this operation requires a choice of reparametrization of the circle. Note the resemblance to the fundamental group: indeed, $\pi_1 X \cong \pi_0 \Omega X$. The group structure follows from what is arguably the most important result in basic category theory, the Yoneda lemma.



A composition of operations in the little 2-disks operad (left), and a depiction of three little 3-disks embedding into a larger 3-disk (right).

If G is a group (or even an ∞-group), one can build a contractible space, called EG, on which G acts freely. The quotient space $BG = EG/G$ then fits into a fibration sequence $G \to EG \to BG$. Since $\pi_n EG \cong 0$ for all n, we have $\pi_{n+1} BG \cong \pi_n G$, so that $G \simeq \Omega BG$.

From this point of view, n-fold *loop spaces* $X_0 \simeq \Omega X_1 \simeq \Omega^2 X_2 \simeq \cdots \simeq \Omega^n X_n$ correspond to spaces with n operations which commute "up to $S^{n-1} \subset \mathbb{R}^n$". As $n \to \infty$, the space S^{n-1} becomes contractible, so that an infinite loop space is the higher categorical incarnation of a commutative group. The precise relation between these commutative ∞-groups and spectra is as follows: any commutative ∞-group G naturally defines a spectrum $\{G \simeq \Omega BG \simeq \Omega^2 B^2 G \cdots\}$, and a spectrum $X = \{X_0 \simeq \Omega X_1 \simeq \Omega^2 X_2 \cdots\}$ is in the image of this functor if and only if $\pi_m X_n = 0$ for $m < n$.

Classically, a commutative group is the same thing as a $\mathbb{Z}$-module. Higher categorically, $\mathbb{Z}$ is no longer the unit of the tensor product on commutative ∞-groups (or spectra), as there is a more initial ring, denoted $\mathbb{S}$, and called the sphere spectrum. (This is not a classical ring; rather, it has nontrivial higher homotopy groups, the calculation of which is a major open problem in algebraic topology.) The reason for this, ignoring inverses for the moment, is that the groupoid $\mathcal{F}in$ of finite sets and bijections is the free commutative monoid on one generator, and not $\mathbb{N}$. While the *group completion* of $\mathbb{N}$ is $\mathbb{Z}$, the group completion of $\mathcal{F}in$ is $\mathbb{S}$. The reason for this seemingly mysterious terminology is that the underlying space of $\mathbb{S}$ is the union

$$\mathbb{S} \simeq \mathrm{colim}_{n\to\infty} \mathrm{Map}_*(S^n, S^n)$$

of the spaces of (pointed) self-maps of the n-sphere S^n.

As envisioned by Waldhausen, one can do brave new algebra over the sphere, or, more generally, a commutative ring spectrum, just as classically one can do ordinary algebra over the integers or some other base commutative ring. This has led to a paradigm shift in several areas of mathematics. For instance, Waldhausen showed that if M is a pointed manifold, then ΩM is an ∞-group, and the K-theory of the group-ring spectrum $K(\mathbb{S}[\Omega M])$ (also obtained by a group completion process) is related to the diffeomorphism groups of M. In fact, the cohomology theory represented by the sphere spectrum $\mathbb{S}$ is *framed* cobordism, another mysterious connection between algebra and geometry. Another derived algebraic invariant of a ring R, called its topological Hochschild homology $\mathrm{THH}(R)$, appears in number theory and arithmetic algebraic geometry in several surprising ways. This is an active research area, and features prominently in this semester's program and its affiliated workshops.

Focus on the Scientist: David Gepner

David Gepner is an Eisenbud Professor in this semester's program on Higher Categories and Categorification. He has made dynamic contributions in and around higher categories, with work ranging from higher algebraic K-theory and derived algebraic geometry, through stable, equivariant, and motivic homotopy theory, to the foundations of ∞-category theory. An enthusiastic collaborator with seventeen coauthors and counting, Gepner has emerged as one of the strongest and most versatile homotopy theorists of his generation, with a lengthy list of highly-cited papers on a broad range of topics.

David Gepner

David grew up in the western suburbs of Chicago where he lived until attending Reed College. He earned his Ph.D. in 2006 from the University of Illinois at Urbana-Champaign. He has held faculty positions at the University of Sheffield, the University of Illinois at Chicago, Universität Regensburg, Purdue University, and the University of Melbourne, where he is currently based.

David's most highly-cited work includes "A universal characterization of higher algebraic K-theory" with Andrew Blumberg and Gonçalo Tabuada; a pair of famous papers with Matt Ando, Andrew Blumberg, Mike Hopkins, and Charles Rezk on Thom spectra; and a foundational paper, "Enriched ∞-categories via non-symmetric ∞-operads," joint with Rune Haugseng, who is also in residence at MSRI this semester.

While it's not possible to convey the depth and originality of David's work in this short space, a recent paper, "∞-Operads as analytic monads," with Rune Haugseng and Joachim Kock, highlights one of his myriad contributions to the foundations of higher category theory. The paper addresses classical operads like the "little k-discs operad" and the monad endofunctors that are built from an operad's composition maps. Gepner, Haugseng, and Kock show that when operads and monads are redeveloped in the less rigid setting of infinite-dimensional category theory, an ∞-*operad* can in fact be recovered from its corresponding endofunctor and those endofunctors that arise from ∞-operads are exactly those that define *analytic* ∞-*monads* (Cartesian monads whose underlying endofunctors preserve sifted colimits and weakly contractible limits).

David brings the same spirit of exploration and collaboration to his extracurricular interests, as a pianist who enjoys dabbling in all genres of music. He is grateful to spend yet another semester at MSRI, and his colleagues are all grateful for the opportunity to work with and learn from him!

— Emily Riehl

Call for Proposals

All proposals can be submitted to the Director or Deputy Director or any member of the Scientific Advisory Committee with a copy to proposals@msri.org. For detailed information, please see the website msri.org/proposals.

Thematic Programs

The Scientific Advisory Committee (SAC) of the Institute meets in January, May, and November each year to consider letters of intent, pre-proposals, and proposals for programs. The deadlines to submit proposals of any kind for review by the SAC are **March 1**, **October 1**, and **December 1**. Successful proposals are usually developed from the pre-proposal in a collaborative process between the proposers, the Directorate, and the SAC, and may be considered at more than one meeting of the SAC before selection. For complete details, see tinyurl.com/msri-progprop.

Hot Topics Workshops

Each year MSRI runs a week-long workshop on some area of intense mathematical activity chosen the previous fall. Proposals should be received by **March 1**, **October 1**, and **December 1** for review at the upcoming SAC meeting. See tinyurl.com/msri-htw.

Summer Graduate Schools

Every summer MSRI organizes several two-week long summer graduate workshops, most of which are held at MSRI. Proposals must be submitted by **March 1**, **October 1**, and **December 1** for review at the upcoming SAC meeting. See tinyurl.com/msri-sgs.

Call for Membership

MSRI invites membership applications for the 2021–22 academic year in these positions:

Research Professors	by October 1, 2020
Research Members	by December 1, 2020
Postdoctoral Fellows	by December 1, 2020

In the academic year 2021–22, the research programs are:

Universality and Integrability in Random Matrix Theory and Interacting Particle Systems
Aug 16 – Dec 17, 2021
Organized by Ivan Corwin, Percy Deift, Ioana Dumitriu, Alice Guionnet, Alexander Its, Herbert Spohn, Horng-Tzer Yau

The Analysis and Geometry of Random Spaces
Jan 18 – May 27, 2022
Organized by Mario Bonk, Joan Lind, Steffen Rohde, Eero Saksman, Fredrik Viklund, Jang-Mei Wu

Complex Dynamics: From Special Families to Natural Generalizations in One and Several Variables
Jan 18 – May 27, 2022
Organized by Sarah Koch, Jasmin Raissy, Dierk Schleicher, Mitsuhiro Shishikura, Dylan Thurston

MSRI uses **MathJobs** to process applications for its positions. Interested candidates must apply online at mathjobs.org after August 1, 2020. For more information about any of the programs, please see msri.org/programs.

Named Postdocs — Spring 2020

Berlekamp

Nicolle González is the Berlekamp Postoctoral Fellow in the Higher Categories and Categorification program at MSRI. Nicolle grew up on the border between Venezuela and Colombia. She completed her undergraduate studies at the University of Oregon and did her Ph.D. at the University of Southern California under the supervision of Aaron Lauda and Sami Assaf. One important result of her thesis is a categorification of the famed Boson–Fermion correspondence. She is now a University of California Presidential Postdoctoral Fellow at UCLA, under the mentorship of Raphaël Rouquier. Nicolle's principal research interests lie in the categorification of structures arising in representation theory, topology, and combinatorics, in particular those related to actions of Heisenberg algebras and the theory of Macdonald polynomials. *The Berlekamp Postdoctoral Fellowship was established in 2014 by a group of Elwyn Berlekamp's friends, colleagues, and former students whose lives he touched in many ways. He was well known for his algorithms in coding theory, important contributions to game theory, and his love of mathematical puzzles.*

Stephen Della Pietra

Colleen Delaney is the Stephen Della Pietra Postdoctoral Fellow in the Quantum Symmetries program. Colleen was an undergraduate physics major in Ricketts Hovse at Caltech before pursuing a Ph.D. in mathematics with Zhenghan Wang at UC Santa Barbara as an NSF Graduate Research Fellow and Microsoft Station Q Graduate Research Fellow. After the Quantum Symmetries program they will resume their postdoctoral position at Indiana University with Julia Plavnik and Noah Snyder as an NSF Mathematical Sciences Postdoctoral Research Fellow. Colleen's research interests lie in algebraic and topological aspects of quantum field theory, most recently in applying fusion categories to better understand the physics of quasiparticles and defects in topologically ordered quantum phases of matter in two spatial dimensions. They also work with computational approaches to quantum invariants of knots and links and their categorifications to homology theories. Colleen requested not to have a photo run with this profile. *The Stephen Della Pietra fellowship was established in 2017 by the Della Pietra Family Foundation. Stephen is a partner at Renaissance Technologies, a board member of the Simons Center for Geometry and Physics, and treasurer of the National Museum of Mathematics in New York.*

Huneke

Cris Negron is the Huneke Postdoctoral Fellow in this semester's Quantum Symmetries program. Cris received his Ph.D. from the University of Washington in 2015, under the supervision of James Zhang. He was an NSF postdoc at Louisiana State University, and a CLE Moore instructor at the Massachusetts Institute of Technology. He recently joined the faculty at the University of North Carolina, Chapel Hill, as an assistant professor. In his earlier works, Cris studied derived invariants for near-geometric objects such as orbifolds and Azumaya algebras. More recently, he has studied exotic quantum groups and their related logarithmic conformal field theories. He has also worked on emerging geometric approaches to studies of finite tensor categories. *The Huneke Postdoctoral Fellowship is funded by a generous endowment from Professor Craig Huneke, who is internationally recognized for his work in commutative algebra and algebraic geometry.*

Strauch

David Reutter is this semester's Strauch Postdoctoral Fellow in both of this Spring's programs at MSRI, Higher Categories and Categorification *and* Quantum Symmetries. David earned his bachelor's and master's degrees in physics from the ETH Zürich and completed Part 3 of the Mathematics Tripos at Cambridge before completing his Oxford D.Phil. in Computer Science under the direction of Jamie Vicary in 2019. He is currently a postdoctoral fellow at the Max Planck Institute for Mathematics under the mentorship of Peter Teichner, who is also in residence at MSRI this semester. In addition to his D.Phil. dissertation, David is the author of nine papers in quantum information theory and mathematics on the boundary of higher category theory and topology, most recently settling in the negative the quarter-century old question of whether Crane–Yetter theory, constructed from what is now called a non-modular braided fusion category, could detect exotic smooth structures. David is very involved in public mathematical outreach, including the development of a public workshop in which participants explore quantum concepts using hand-held q-bit simulators. *The Strauch Fellowship is funded by a generous annual gift from Roger Strauch, Chairman of The Roda Group. He is a member of the Engineering Dean's College Advisory Boards of UC Berkeley and Cornell University, and is also currently the chair of MSRI's Board of Trustees.*

Viterbi

Alexander Campbell is this semester's Viterbi Postdoctoral Fellow and a member of the Higher Categories and Categorification program. Alexander is a category theorist from Sydney, Australia. From 2009–12, he was an undergraduate at the University of Sydney, and from 2013–19, he was a member of the Centre of Australian Category Theory at Macquarie University, first as a Ph.D. student under the supervision of Ross Street, and subsequently as a postdoctoral research fellow with Richard Garner. Alexander's research interests are in category theory, especially higher and enriched category theory, and categorical aspects of homotopy theory. In 2019, he solved two longstanding open problems in the theory of quasi-categories (aka, infinity-categories) posed by André Joyal. While at MSRI, Alexander is working on a number of projects concerning higher infinity-categories jointly with other members of the Higher Categories program. *The Viterbi postdoctoral fellowship is funded by a generous endowment from Dr. Andrew Viterbi, well known as the co-inventor of Code Division Multiple Access based digital cellular technology and the Viterbi decoding algorithm, used in many digital communication systems.*

Mathical Continues to Grow in Stature and Reach

A new collection of Mathical titles at the ELG Rapid Rehousing Center, a family homeless shelter in the Bronx. Image courtesy of the Books for Kids Foundation.

Core and Founding Partners

Mathical is hitting a growth spurt, with greater recognition among publishers, librarians, teachers, and parents. We are grateful to our core partners. One is the **Children's Book Council**, which helps us get the word out to publishers; others are the **National Council of Teachers of English** (NCTE) and the **National Council of Teachers of Mathematics** (NCTM), for opening doors with educators.

NCTM teamed up with MSRI, and together we were poised to have eight Mathical authors at the NCTM Centennial conference in Chicago, in April 2020. Many components of the event have been postponed to Fall 2020, and we hope that Mathical content will continue to be among them.

We are also grateful to the **Simons Foundation** for its founding support of the prize, and for support along the way from many including the **Heising-Simons Foundation** and individual donors including former MSRI board chair **Roger Strauch**, who believed strongly in the mission of the prize from the start.

For the newest batch of Mathical winners (see inset; their covers are shown on this issue's cover), we took the announcement event online in March — with great success! — and you can find the video at tinyurl.com/mathicalCIME2020.

Reaching Low-Income Schools and Children

MSRI has a new partnership with the **School Library Journal**, a savvy group who are helping us reach low-income kids at Title I schools by administering a new joint program, the **Mathical Book Prize Collection Development Awards**. With their help, up to 25 schools nationwide will receive $700 awards this spring to purchase large sections of the Mathical List for their school libraries.

MSRI is working newly with the **Books for Kids Foundation** to distribute Mathical titles nationally this spring to libraries serving low-income preschool children. In addition, the Books for Kids Foundation is placing a range of Mathical titles in newly established family libraries in homeless shelters in New York City.

Both of these new partnerships are made possible through the generous support of the **Patrick J. McGovern Foundation**.

MSRI also continues our longstanding partnership with **First Book**, which provides high-quality new books and educational resources to low-income children in programs and schools, reaching an estimated one in three children in need through its ever-growing national network. Our work with them focuses on kids in need in the Bay Area, and is generously supported by the **Firedoll Foundation**.

A Growing Network of Partners

Mathical's other partnerships are almost too many to mention, but we will try to remember them all:

The **Association of Children's Museums** is hosting a webinar for museum educators this month to learn more about Mathical and early math learning; **Bring Me a Book**, a Bay Area nonprofit, is working with MSRI to exchange resources and information; **Bookelicious** is a new app for children's book pairings that will launch in Q2 of 2020; **Development and Research in Early Math Education** (DREME, out of Stanford) is working together with MSRI to create more reading guides for Mathical titles to make it easier for parents and non-math teachers to enjoy teaching the books; we are working with the **Internet Archive** to exploring getting Mathical titles online in a new way; **Reach Out and Read** shares Mathical information with physicians in early child well visits; **Teaching Books** features a Mathical collection that serves as a pointer for educators looking to stock their classrooms.

2020 Mathical Book Prize Winners

MSRI's Mathical Book Prize recognizes outstanding fiction and literary nonfiction for youth aged 2–18. The prize is selected annually by a committee of pre-K-12 teachers, librarians, mathematicians, early childhood experts, and others. This year's winners are:

Pre-K: *One Fox: A Counting Book Thriller* by Kate Read
Grades K–2: *Pigeon Math* by Asia Citro
Grades 3–5: *Solving for M* by Jennifer Swender
Grades 9–12: *Slay* by Brittney Morris

Have a book you hope we'll consider? Everyone can submit your favorite math-inspiring titles for possible inclusion as a Mathical Honor Book or in the Mathical Hall of Fame.

The Mathical Book Prize is awarded by MSRI in partnership with the National Council of Teachers of English and the National Council of Teachers of Mathematics, and in coordination with the Children's Book Council. The Mathical list is intended as a resource for educators, parents, librarians, children, and teens. Download the list at mathicalbooks.org.

Quantum Subgroups: Back to the Future

Terry Gannon and Cris Negron

In order to make sense of modern physics, mathematicians are being challenged to generalize their classical notions. Quantum symmetries, the theme of this semester's program, is what is evolving from groups and Lie theory. It covers a wide range: quantum computers, string theory, knots, subfactors, topological quantum field theory, Hopf algebras, tensor categories, ... The following article focuses on one influential storyline running through the history of the subject, which has been shaped by several participants of this program.

Modular Invariants and the Classification of Conformal Field Theories

The language of modern physics is quantum field theory, and understanding it is a major challenge for mathematics. The most accessible class of quantum field theories are the conformal field theories (CFTs), as they are especially simple (only one space and one time dimension), and especially symmetrical (the conformal transformations are the maximal space-time symmetries allowed by relativity). CFTs arise for instance in perturbative string theory and condensed matter.

In 1986 Cappelli, Itzykson, and Zuber addressed the classification of the possible (rational) CFTs associated to the Lie algebra $sl(2)$. They reduced it to the following problem:

For each integer $\kappa \geqslant 3$, define matrices

$$S_{a,b} = \sqrt{\frac{2}{\kappa}}\sin\left(\frac{\pi ab}{\kappa}\right),\ T_{a,b} = \delta_{a,b}e^{-\pi i/4}e^{\pi i a^2/2\kappa},$$

for $1 \leqslant a, b < \kappa$. Find all matrices M such that

(i) $SM = MS$ and $TM = MT$;

(ii) $M_{a,b} \in \mathbb{Z}_{\geqslant 0}$;

(iii) $M_{1,1} = 1$.

Any such matrix M is called a modular invariant for $sl(2)$. Equivalently, we can write M in the form

$$\mathcal{Z} = \sum_{a,b=1}^{\kappa-1} M_{a,b}\chi_a\chi_b^*$$

for formal variables χ_a, χ_b^*. For example, the identity $M = I$ is a modular invariant, corresponding to $\mathcal{Z} = \sum_{a=1}^{\kappa-1}\chi_a\chi_a^* = \sum_{a=1}^{\kappa-1}|\chi_a|^2$.

Intriguingly, the list of $sl(2)$ modular invariants fits into an A-D-E pattern: two infinite sequences of graphs A_m and D_n, as well as three exceptionals called E_6, E_7, E_8. Some of these are displayed in the figure in the next column. A-D-E is a common answer to seemingly different questions in math and mathematical physics. They classify the simply-laced simple Lie algebras, and also parametrize the finite subgroups of the matrix group $SL_2(\mathbb{C})$. The key property of these graphs, which is responsible for their ubiquity, is that their adjacency matrices have largest eigenvalue < 2.

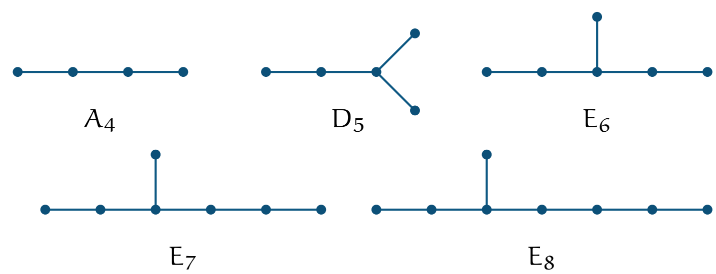
Each solution M can be assigned an A-D-E graph which shares some of its properties: κ is the so-called Coxeter number of the graph, and the values of a with $M_{a,a} \neq 0$ are its so-called exponents. For example, $M = I$ is assigned the $A_{\kappa-1}$ graph, while D_5 and E_8 are assigned to

$$\mathcal{Z} = |\chi_1|^2 + |\chi_3|^2 + |\chi_4|^2 + |\chi_5|^2 + |\chi_7|^2 + \chi_2\chi_6^* + \chi_6\chi_2^*,$$
$$\mathcal{Z} = |\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}|^2 + |\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}|^2,$$

which work for $\kappa = 8$ and 30, respectively.

A CFT itself consists of two halves, called vertex operator algebras (VOAs), and a matching of the irreducible representations of those VOAs. A VOA is an infinite-dimensional space with infinitely many bilinear products — a complicated enough beast that its formal definition is best avoided. κ parametrizes the different $sl(2)$-type VOAs $\mathcal{V}(sl(2), \kappa)$. The S and T matrices define a representation of the modular group $SL_2(\mathbb{Z})$ fundamental to the theory. The modular invariant explains how to extend $\mathcal{V}(sl(2), \kappa)$ to get those two halves and gives the matching.

For example, the $A_{\kappa-1}$ modular invariants involve trivial VOA extensions and the identity matching. For E_8, both extensions are to the VOA $\mathcal{V}(G_2, 5)$, with trivial matching. The D_5 modular invariant has trivial extensions but nontrivial matching.



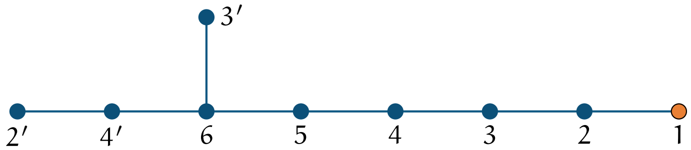
The A-D-E diagrams.

Nim-Reps and Boundary CFT

To deepen the connection between graphs and modular invariants, we need to give meaning to the edges. One classical story, McKay's association of graphs to finite subgroups of $SL_2(\mathbb{C})$, suggests how. Consider the symmetry group of the icosahedron for concreteness. This size-120 subgroup has nine irreducible representations $\rho_1, \rho_2, \rho_{2'}, \ldots, \rho_6$ (labelled by dimension), which form the vertices. Call ρ_2 the defining representation in $SL_2(\mathbb{C})$. Draw a directed edge from vertex a to b if ρ_b occurs in the decomposition of the tensor product $\rho_2 \otimes \rho_a$. For example, $\rho_2 \otimes \rho_2 \cong \rho_1 \oplus \rho_3$, so we draw directed edges from vertex 2 to vertices 1 and 3. The arrows on an edge directed in both directions are erased. The resulting graph is extended E_8. The largest eigenvalue of this graph (namely, 2) is the dimension of ρ_2.

Something similar happens in CFT. The VOA representations ($\chi_1, \ldots, \chi_{\kappa-1}$ in our case) act on the boundary states of the theory. This action is called a nim-rep, short for "nonnegative integer matrix representation." The number of indecomposable boundary states equals the trace of the modular invariant. If we draw the

McKay graph for this nim-rep, with vertices labelled by the boundary states and edges describing how χ_2 acts, we recover precisely the A-D-E graph assigned to that modular invariant. The largest eigenvalue of the graph is the quantum-dimension $2\cos(\pi/\kappa)$ of representation χ_2. The A-D-E classification of $\mathfrak{sl}(2)$ CFT is a sort of quantum-deformation of the A-D-E classification of subgroups of $SL_2(\mathbb{C})$, hence the term, *quantum subgroups*.



extended E_8

McKay graph for binary icosahedral group.

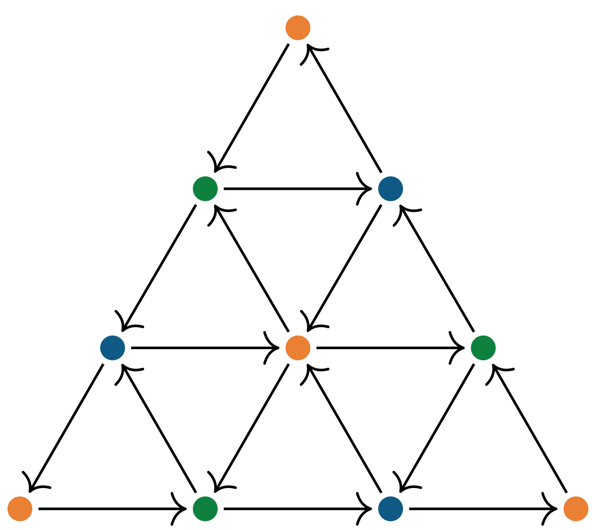
Galois and Higher Rank

Because the $\mathfrak{sl}(2)$ classification is so interesting, we should consider the modular invariants for other Lie algebras $\mathfrak{g}$. Again, for each κ, we get another VOA $\mathcal{V}(\mathfrak{g},\kappa)$, with its own S and T matrices. Unfortunately, the method of Cappelli–Itzykson–Zuber completely fails even for $\mathfrak{sl}(3)$. New ideas were needed to explore the modular invariant classifications for higher rank $\mathfrak{g}$.

It was discovered within this context that the S and T matrices always have a very special *Galois symmetry*. For $\mathfrak{sl}(2)$, this implies the following condition: $M_{1,a} \neq 0$ requires

$$\cos\left(\pi\ell\frac{a-1}{\kappa}\right) > \cos\left(\pi\ell\frac{a+1}{\kappa}\right)$$

for all ℓ coprime to κ. Using this condition, the proof of the classification for $\mathfrak{sl}(2)$ can be shortened by a factor of over 15. More importantly, this Galois symmetry led in 1994 to the classification of $\mathfrak{sl}(3)$ modular invariants. These don't fall into an A-D-E pattern, nor seem connected to finite subgroups of $SL_3(\mathbb{C})$, but have mysterious relations with simple factors in the Jacobians of Fermat curves. A nim-rep graph for $\mathfrak{sl}(3)$ is given in the figure below.



Nim-rep graph for $M = I$ and $\mathcal{V}(\mathfrak{sl}(3),6)$.

For higher rank Lie algebras, the Galois condition remains powerful, but is hard to use. So for a while, the modular invariant classifications stopped at $\mathfrak{sl}(3)$. The hard part was controlling the possible VOA extensions of $\mathcal{V}(\mathfrak{g},\kappa)$. New ideas again were needed.

Operator Algebra Takes Over

Already in 1990, Vaughan Jones in his ICM plenary talk mentioned the Cappelli–Itzykson–Zuber result as hinting at deep connections between subfactors and CFT. The next decade or so fleshed out a rich structure underlying modular invariants and their associated nim-reps. VOA representations χ_a and CFT boundary states were modelled by morphisms between von Neumann algebras, so that tensor product simplifies to composition. But perhaps the most important discovery was *alpha-induction*.

Classically, group representations restrict to subgroup representations. This preserves dimension and respects tensor product. In group theory, restriction has an adjoint, called induction, which canonically lifts a representation of the subgroup to that of the group. Implicit in modular invariants are VOA extensions; is there a VOA analogue of induction?

The answer the subfactor community discovered was *yes*, though with a difference. The target of this new induction isn't the representations of the larger VOA, but a larger, looser class called twisted representations. Here induction, and not restriction, preserves dimensions and respects tensor product.

For concreteness consider the E_8 modular invariant. The vertices correspond to the twisted representations of the VOA $\mathcal{V}(G_2,5)$. The modular invariant only sees the true representations ("11" and "12"). Beside each node are the restrictions to $\mathcal{V}(\mathfrak{sl}(2),30)$. Representation χ_2 of $\mathcal{V}(\mathfrak{sl}(2),30)$ induces to the twisted representation τ_1, so we find that, for example, $\tau_1' \otimes \tau_1' \cong (11) \oplus \tau_1'$.

The Categorical Interpretation

We now know, based on a number of fundamental works from the early 2000s, that the representation theory of the VOA $\mathcal{V}(\mathfrak{g},\kappa)$ forms a modular tensor category (MTC), and the twisted representations form a fusion category. The nim-rep and modular invariant are captured by a module category for the MTC. (A module category over an MTC is the straightforward categorification of the notion of a module over a commutative ring.) The matching mentioned earlier is a braided tensor equivalence between the MTCs of the two extended VOAs. Alpha-induction is a tensor functor (adjoint to restriction) from the MTC of say $\mathcal{V}(\mathfrak{sl}(2),30)$ to the fusion category of twisted representations of $\mathcal{V}(G_2,5)$. This reinterpretation is important in that it makes the deep ideas of the subfactor community much more widely accessible. We now know that the full structure of a CFT can be recovered from the initial VOA and the module category.

It took 20 years, but we now know the question Cappelli–Itzykson–Zuber were asking: *Find all module categories of the* $\mathcal{V}(\mathfrak{sl}(2),\kappa)$ *modular tensor category.*

The Endgame: Induction Meets Galois

The modular invariant classifications have essentially been stalled for 25 years at $\mathfrak{sl}(3)$, despite the fact that large swaths of nim-reps

Focus on the Scientist: Dmitri Nikshych

Dmitri Nikshych is a world-renowned researcher in quantum symmetries. He has made substantial contributions to weak Hopf algebras, quantum groups, and tensor categories, most notably in his joint paper with Pavel Etingof and Victor Ostrik, "On Fusion Categories" (Annals, 2005).

Dmitri Nikshych

His recent book, *Tensor Categories* (AMS, 2015), co-written with Etingof, Ostrik, and Shlomo Gelaki, is now widely considered to be "The Book" for those (interested) in quantum symmetries. Dmitri's work is exciting, deep, and innovative, and is greatly appreciated by algebraists, analysts, topologists, and physicists around the globe.

Dmitri was raised in Kiev, Ukraine. His parents fostered his mathematical interests at a young age, and with this support, he was admitted into a competitive high school that specializes in mathematics. He then enrolled at the Kiev Polytechnic Institute (KPI) to study applied math, and received great training in functional analysis during this time. This drew him to the precision and beauty of theoretical mathematics, especially to quantum groups via the generous guidance of Leonid Vainerman.

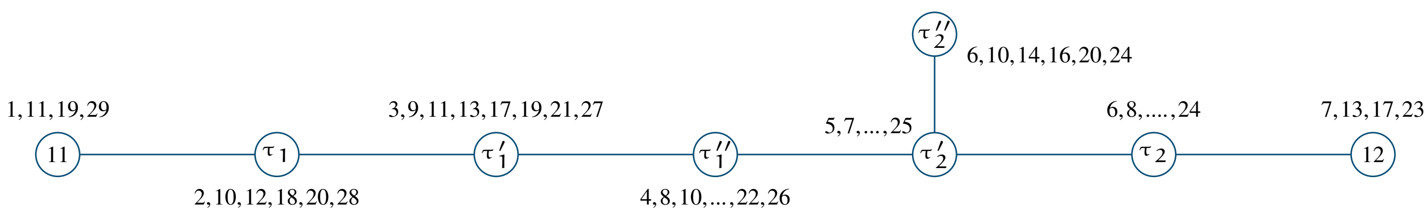
After KPI, Dmitri began his graduate studies at UCLA in 1997 with advisor Edward Effros. During his second year, Effros pointed out to Dmitri a course on quantum groups that Etingof was giving at MIT, and Dmitri soon wrote Etingof to ask if he could visit to attend the course. His wish was granted and he ended up sticking around MIT for nearly two years! Both Etingof and Ostrik (who was also at MIT at the time) introduced Dmitri to many new ideas, and this sparked a transformative collaboration and continuing friendship.

Dmitri defended his thesis at UCLA in 2001 and joined the University of New Hampshire as an assistant professor that year. With having a child on the way, he was grateful for the terms and stability of his new position.

A year later, Dmitri's ground-breaking preprint with Etingof and Ostrik on fusion categories appeared. Other notable works appearing since then include "On braided fusion categories I" (Selecta, 2010) joint with Gelaki, Ostrik, and Vladimir Drinfeld, and "The Witt group of non-degenerate braided fusion categories" (Crelle, 2013) joint with Ostrik, Alexei Davydov, and Michael Müger.

In New Hampshire, Dmitri lives with his wife, three kids, and several pets including a chicken named Nightfury. He enjoys spending time with his family, reading history books, and picking mushrooms in his spare time.

— Chelsea Walton



The E_8 example revisited.

(indeed, module categories) have been worked out for a few other Lie algebras. However, recent breakthroughs, in part from two postdocs in our program, promise a glut of new module category classifications in the near future.

Recall that classifying the module categories requires two things: determining the possible VOA extensions of $\mathcal{V}(\mathfrak{g},\kappa)$ and determining all braided tensor equivalences between the MTC of those extensions.

Schopieray used alpha-induction to find an explicit bound on κ for $\mathfrak{g} = sp(4)$ and G_2, beyond which any VOA extension of $\mathcal{V}(\mathfrak{g},\kappa)$ would have to be of an especially nice type. In principle this bound could be computed for any Lie algebra, though it grows exponentially with the rank. However, it has since been shown that Galois symmetry can be combined with alpha-induction to come up with vastly smaller bounds, growing only cubically with the rank. For example, for $\mathcal{V}(G_2,\kappa)$ Schopieray's κ-bound is 18 million, while Galois and alpha-induction reduce it to 129.

Using these bounds, the classification of all possible extensions of $\mathcal{V}(\mathfrak{g},\kappa)$, for all Lie algebras $\mathfrak{g}$ of small rank (for example, $\leqslant 8$) should be imminent. The result in small rank is that there are very few extensions which are not of the especially nice type.

Furthermore, Edie-Michell has recently worked out the autoequivalences of the MTC for $\mathcal{V}(\mathfrak{g},\kappa)$. The final step in the quantum subgroup (aka, module category) classifications for all Lie algebras of small rank, will be the application of Edie-Michell's methods to the especially nice extensions.

Quantum subgroups for $sl(2)$ fall into an A-D-E pattern. Those for $sl(3)$ seem directly related to Jacobians of Fermat curves. It is exciting to speculate what these relate to in higher rank! We should soon know!

Puzzles Column

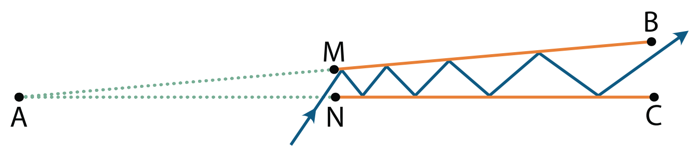
Joe P. Buhler and Tanya Khovanova

1. Find all possible positive integers a, b, c, and d such that $ab = c + d$ and $cd = a + b$.

2. Let C be the surface of a cube in 3-space. What is the largest n such that there is a regular n-gon P, not lying entirely on one face of C, such that all P's vertices are on C?

3. Here is a different sort of hat problem. There are n hats, each with a different color. These hats are placed on the heads of n sages. All of the sages know all the colors: their own hat and everyone else's hats. A referee then announces the "correct" hat color that should be on the head of each sage.

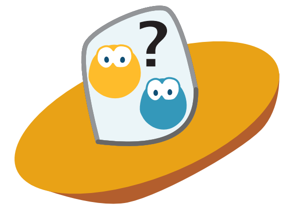
The sages are then allowed "swap" sessions: in one session disjoint pairs of sages are allowed to interchange their hats. Can the sages fully correct their hat colors in two swap sessions?



4. Line segments AB and AC have equal length and have an angle of one degree at A. Let M be the midpoint of AB and N be the midpoint of AC. Erase AM and AN, so that the remaining segments MB and NC are disjoint and at an angle of one degree. (This is illustrated, for a much larger angle, in the figure.) We think of these line segments as mirrors in that they reflect light perfectly in the usual way.

A light ray in the plane containing ABC enters quadrilateral $MBCN$ at the thin end, between M and N. Then it bounces back and forth until it exits at the thick end, between B and C. What is the maximum possible number of reflections that it can make?

Comment: This appeared in a 2001 book of 50 mathematical puzzles, edited by G. Cohen, and we first heard about it from Stan Wagon and Dan Velleman.



5. A race of aliens tours our galaxy looking for planets with edible humanoids. When they visit a planet they take an instantaneous census of all humanoids, and classify each as either edible or inedible. If the number of pairs of edible humanoids is strictly bigger than one-half of the total number of pairs of humanoids, then all of its edibles are promptly hunted and eaten.

Earth was visited by these aliens sometime in the last century. We (or, at least those of us that were judged to be edible) escaped by the narrowest possible margin. What year did the aliens visit?

Comment: This problem is due to Steve Silverman.

6. N people are randomly placed on a line segment. Each person turns to face their closest neighbor (with probability 1, all the distances are distinct, so this is sufficiently well-defined). What is the expected number of "lonely people" who are looking at a neighbor who is not looking at them?

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Nov 30–Dec 4, 2020: *Structure and Randomness in Locally Symmetric Spaces*

Dec 7–11, 2020: *Topical Workshop: Decidability, Definability and Computability in Number Theory*

Summer Graduate Schools

Jun 8–19, 2020: *Combinatorial and DG-Algebra Techniques for Free Resolutions (Tianjin, China)*

Jun 15–26, 2020: *Geometric Flows (Athens, Greece)*

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To find more information about any of these workshops, as well as a full list of all upcoming workshops and programs, please see msri.org/workshops and msri.org/programs.

2019 Annual Report

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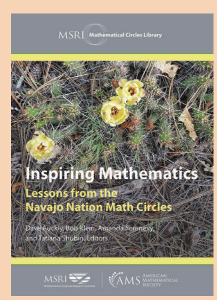

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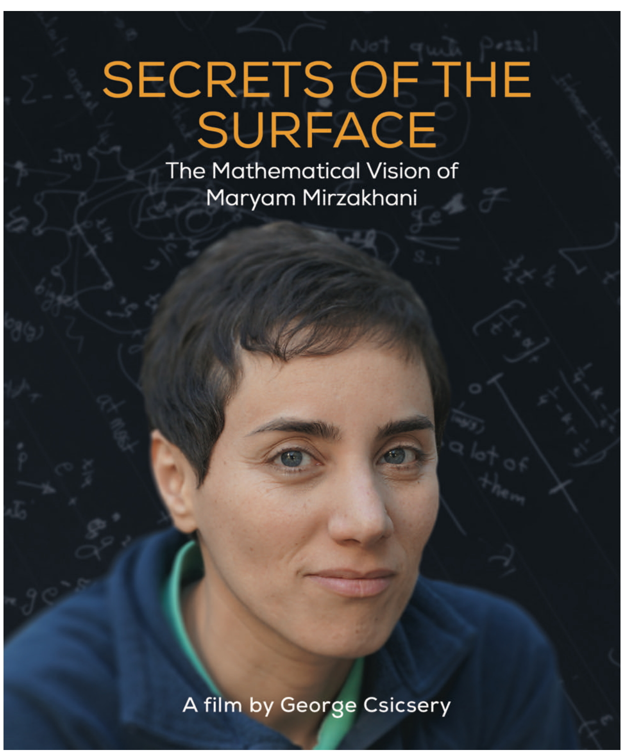

MSRI's Newest Documentary Shares the Life and Work of Maryam Mirzakhani

Filmed in Canada, Iran, and the United States, *Secrets of the Surface: The Mathematical Vision of Maryam Mirzakhani* (2020) examines the life and mathematical work of Maryam Mirzakhani, an Iranian immigrant to the United States who became a superstar in her field. In 2014, she was both the first woman and the first Iranian to be honored with one of mathematics' highest prizes, the Fields Medal.

Maryam's contributions are explained in the film by leading mathematicians and illustrated by animated sequences. Her mathematical colleagues from around the world, as well as former teachers, classmates, and students in Iran today, convey the deep impact of her achievements. The path of her education, success on Iran's Math Olympiad team, and her brilliant work, make Mirzakhani an ideal role model for girls looking toward careers in science and mathematics.

Secrets of the Surface is now available at www.zalafilms.com/secrets. If you would like to arrange a screening of the film at your institution, contact the film's director, George Csicsery, at geocsi@zalafilms.com.