Puzzles Column

Joe P. Buhler and Tanya Khovanova

Many mathematicians love brain teasers. At David Eisenbud's strong suggestion, Elwyn Berlekamp and I (JB) started writing the Emissary Puzzles Column when I arrived as Deputy Director in the fall of 1999. The idea was to capture contagious bits of mathematics brought to MSRI by its steady stream of visitors.

This has been a vivid and delightful experience for me, and I have many fond memories of discussions, in person and in email, about which problems to include, how to solve them, and (especially) how to phrase them. Elwyn was never shy about criticizing anything that he viewed as imprecise, inelegant, or unartfully phrased, but he was also full of delight at new and interesting ideas. This interchange continued unabated up to the very last few weeks of his life.

This column is dedicated to the memory of Elwyn, and his love of puzzles. We include a diverse selection of puzzles that evoke Elwyn, in one way or another. One of them is an open question, due to Elwyn, that arose out of a "hat puzzle" in the column; Elwyn would really like someone to solve it!

Mathematicians in hats, a frequent subject of the Puzzles Column. David Eisenbud with Elwyn Berlekamp in October 2015.

Puzzles for Elwyn

1. Prove that the order of any automorphism of a finite group of order $n > 1$ has order less than n.

Comment: DO NOT WORK ON THIS PROBLEM! This was the last problem of the Fall 1999 column. An eminent member of the UC Berkeley mathematics department proposed it and said that it was a bit on the harder side, despite the simplicity of the question. In fact, it is extremely difficult; it can be done by (slightly tedious) enumeration of cases starting with the classification of simple groups, but two solutions in print are both unbelievably intricate and dense (to any non-group-theorist); asking the Emissary reader to solve this was "cruel and unusual." For the next twenty years, we solved all problems before including them. (The alert reader will note that our final problem here contravenes this stricture, albeit overtly.)

2. The decimal representation of 2^{29} has nine distinct digits. Which digit is missing?

Comment: This problem was proposed by the aforementioned UCB mathematician, and we were happy that it was on the other end of the difficulty spectrum. When it was described to Elwyn orally, he instantly said that he didn't like the problem because you could just plug it into a calculator and be done with it. About 10 seconds later, he had an "aha" moment and said that, no, he thought that it was a great problem!

3. Each of the twelve faces of a dodecahedron has a light that is also an on/off button. Pushing the light causes all five of the lights on the adjacent faces to switch state (go from on to off, or the reverse). Prove that any of the 2^{12} positions can be obtained from any other by a suitable sequence of button pushes.

Comment: Elwyn introduced the famous rectangular grid "lights out" puzzle at Bell Labs in the 1960s. The idea has been generalized in many directions; the dodecahedral version above was commercially available more than 20 years ago, and other variants can be found either in game stores, or in math papers. (For example: Martin Kreh (2017), "'Lights Out' and Variants," The American Mathematical Monthly, 124:10, 937-950.)

4. Find three random variables each uniformly randomly distributed on $[0,1]$ such that their sum is constant. That is, find a probability distribution on the intersection of $x+y+z=1$ with the unit cube in 3-space such that the three "coordinate projections" onto $[0,1]$ are uniform.

Comment: Numerous solutions are possible, and this question generated more correspondence and greater diversity of solutions than any other Emissary puzzle has.

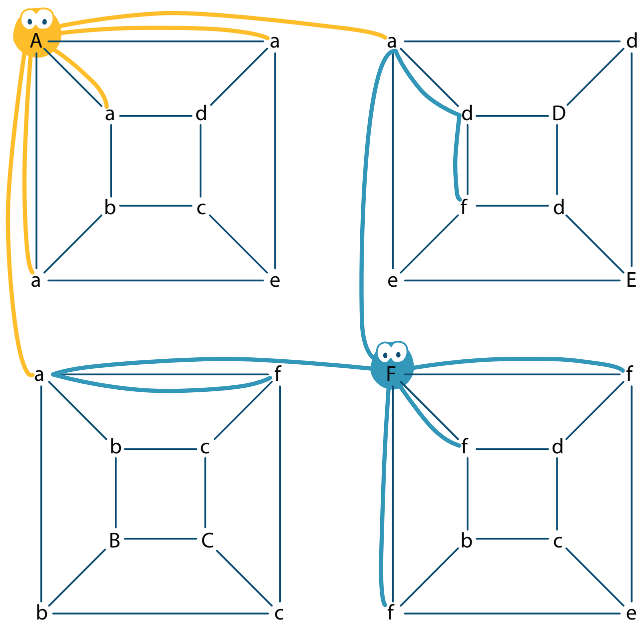
5. (i) Alice and Bob each have \$100 and a biased coin that comes up heads 51% of the time. On a signal, they each start flipping their coins once a minute, betting \$1 on the result of the flip. Alice bets on heads; poor Bob bets on tails. As it happens, they both eventually go broke. In that case, who is more likely to have gone broke first?

(ii) As above, but this time Alice and Bob bet on the result of the same coin flip (say, flipped by a referee). Again, assume both eventually go broke. Who is more likely to have gone broke first?

Comment: Elwyn was always a great fan of Peter Winkler's work, perhaps especially including his "recreational" mathematical problem books. This problem was contributed by Peter for the sake of this column.

6. Let H_n be the n-dimensional hypercube graph, whose vertices are (labeled by) the 2^n binary strings of length n, with edges joining vertices whose labels differ in exactly one coordinate. A *spider* in this graph consists of a central vertex c together with n paths, called legs, that start at c. A set of spiders covers H_n if no legs share an edge (though they are allowed to cross at vertices) and every vertex is either a central vertex of some spider, or an endpoint of a spider leg. Prove that there is a collection of m spiders that cover with exactly

$$m = \lceil \frac{2^n}{n+1} \rceil.$$



Note that this is the minimum possible number, since each spider accounts for $n+1$ points (its center and each endpoint of a leg) and there are 2^n points.

The figure shows a hypercube in five dimensions, where some edges are omitted so as not to clutter the diagram. Namely, every vertex has two extra edges connecting it to its sisters in the horizontally and vertically adjacent 3-cubes. A covering by six spiders is illustrated — the spiders are at the upper case letters, and the endpoints of their legs are at corresponding lower case letters. It is almost a covering by Hamming balls (one example being spider A), except that spider F has one leg of length two and one of length three.

Comment: The "hat problem" in the Fall 2001 column was referenced in a *New York Times* article ("Why Mathematicians Now Care About Their Hat Color," April 10, 2001) and has been cited in several academic papers; it is due to Todd Ebert. An earlier paper by Aspnes, Beigel, Furst, and Rudich had a "voting" problem that has a very similar solution (using perfect Hamming codes). The latter can be expressed as a "majority hats problem," and the truth of the above conjecture, which Elwyn made after he had settled many cases, would imply that the solution to the optimal probability for the voting problem is unexpectedly explicit whereas the size of the solution for the original hat game seems to be unknowable in polynomial time (except, roughly, for $n = 2^k$). All this will be explained in detail in the problem solutions.

Puzzle solutions follow on next page

Emissary Puzzles Column
Solutions, Fall 2019

May 7, 2020

Puzzles for Elwyn

Problem 1. Prove that the order of any automorphism of a finite group of order $n > 1$ has order less than n.

Comment: This is an extraordinarily difficult problem. You were explicitly asked not to work on it . . . A proof can be found inn Horoševskiĭ, M.!V., Automorphisms of finite groups. *Mat. Sb.* (N.S.) 93(135) (1974), 576–587, 630. A more general result asserts that if G has an autormorphism of order larger than $|G|/2$ then G is abelian; this can be found in: Bors, Alexander, Finite groups with an automorphism of large order. *J. Group Theory* 20 (2017), no. 4, 681–717.

Problem 2. The decimal representation of 2^{29} has nine distinct digits. Which digit is missing?

Solution: Let d be the missing digit. Since $2^6 \equiv 1 \bmod 9$, we see that $2^{29} \equiv 1/2 \equiv 5 \bmod 9$. This is the same thing as the digit-sum (mod 9) of 2^{29}. Since the sum of all 10 digits is $45 \equiv 0 \bmod 9$, we have $5 + d \equiv 0 \bmod 9$ so that the missing digit is $d = 4$.

Problem 3. Each of the twelve faces of a dodecahedron has a light that is also an on/off button. Pushing the light causes all five of the lights on the adjacent faces to switch state (go from on to off, or the reverse). Prove that any of the 2^{12} positions can be obtained from any other by a suitable sequence of button pushes.

Solution: The Krebs article in the *Monthly* referenced in the comment in the Emissary gives a general approach to this sort of problem; the key is to showing "full reachability" is to show that a suitable adjacency matrix has full rank. In the dodecahedron case it is easier to note that if the 5 neighbors of a vertex v are touched, then v switches state and all other vertices are unchanged.

Problem 4. Find three random variables each uniformly randomly distributed on [0,1] such that their sum is constant. I.e., find a probability distribution on the intersection of $x+y+z = 3/2$ with the unit cube in 3-space such that the three "coordinate projections" onto $[0, 1]$ are uniform.

Comment/Correction: In the statement of the problem in the *Emissary* the equation of the plane was given as $x + y + z = 1$. In fact, the sum of 3 random variables that are uniform on $[0, 1]$ has mean $3/2$, so if the sum is constant then the sum has to be $3/2$.

Solution: The "trick" solution that was making the rounds among information theorists a few years ago was: Let X be a uniform random variable on $[0, 1]$. If x is in $[0, 1)$ let x' denote the number obtained by writing the ternary expansion of x and incrementing each trit modulo 3. (The reader will surely remember that the ternary expansion of 1 is .2222.) Define random variables Y and Z by giving Y the value x' and Z the value $(x')'$ when X takes the value x.

A slightly more prosaic solution is to pick X uniform on $[0, 1]$, let $Y = X + 1/2 \bmod 1$, and $Z = 1 - 2X \bmod 1$.

Problem 5. *i.* Alice and Bob each have \$100 and a biased coin that comes up heads 51% of the time. On a signal, they each start flipping their coins once a minute, betting \$1 on the result of the flip. Alice bets on heads; poor Bob bets on tails. As it happens, they both eventually go broke. In that case, who is more likely to have gone broke first?

ii. As above, but this time Alice and Bob bet on the result of the same coin flip (say, flipped by a referee). Again, assume both eventually go broke. Who is more likely to have gone broke first?

Solution: Let $p = 0.51$ and $q = 0.49$.

i. Given that they both go broke, they are equally likely to become broke first.

The probability of Bob going broke is 1, while the probability of Alice going broke is $\left(\frac{q}{p}\right)^{100}$.

Consider a string B of tosses where at the end Bob loses. If we flip it swapping heads and tails, we get a string A where Alice loses. String B contains 100 more heads, while A contains 100 more tails. Such strings are in one-to-one correspondence. Each particular sequence of $2k + 100$ tosses with k tails and $k + 100$ heads ending in bancruptcy for Bob has probability $B_k = q^k p^{k+100}$. Correspondingly, each particular sequence of $2k + 100$ tosses with k heads and $k + 100$ tails ending in bankruptcy for Alice, has probability $A_k = p^k q^{k+100}$. The ratio of these is a constant $\frac{A_k}{B_k} = \left(\frac{q}{p}\right)^{100}$, exactly the apriori probability that Alice goes broke.

That means, they will lose at the same time. Moreover, the distribution of each of them being broken after a given number of tosses is the same, conditioned on the fact that Alice goes broke.

ii. Given that they both go broke, Alice is more likely to go broke first.

Again, swapping heads with tails creates a bijection between cases when Alice goes broke first while Bob is second and vice versa. Since in the first case there are 100 more heads and in the second case there are 100 more tails, each instance of the first case is $\left(\frac{q}{p}\right)^{100}$ more probable than the corresponding instance of the second case. Thus, given that they both go broke, Alice is first with probability $\frac{p^{100}}{p^{100}+q^{100}}$, which is slightly more than 98 percent.

Problem 6. Let H_n be the n-dimensional hypercube graph, whose vertices are (labeled by) the 2^n binary strings of length n, with edges joining vertices whose labels differ in exactly one coordinate. A *spider* in this graph consists of a central vertex c together with n paths, called legs, that start at c. A set of spiders covers H_n if no legs share an edge (though they are allowed to cross at vertices) and every vertex is either a central vertex of some spider, or an endpoint of a spider leg. Prove that there is a collection of spiders that cover with exactly

$$m = \lceil \frac{2^n}{n+1} \rceil.$$

Note that this is the minimum possible number, since each spider accounts for $n + 1$ points (its center and edge endpoint of a leg) and there are 2^n points.

Comment: This problem is open! Please let us know if you solve it.