Puzzles Column

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1. Find all possible positive integers \( a, b, c, \) and \( d \) such that \( ab = c + d \) and \( cd = a + b \).

2. Let \( C \) be the surface of a cube in 3-space. What is the largest \( n \) such that there is a regular \( n \)-gon \( P \), not lying entirely on one face of \( C \), such that all \( P \)'s vertices are on \( C \)?

3. Here is a different sort of hat problem. There are \( n \) hats, each with a different color. These hats are placed on the heads of \( n \) sages. All of the sages know all the colors: their own hat and everyone else’s hats. A referee then announces the “correct” hat color that should be on the head of each sage.

   The sages are then allowed “swap” sessions: in one session disjoint pairs of sages are allowed to interchange their hats. Can the sages fully correct their hat colors in two swap sessions?

4. Line segments \( AB \) and \( AC \) have equal length and have an angle of one degree at \( A \). Let \( M \) be the midpoint of \( AB \) and \( N \) be the midpoint of \( AC \). Erase \( AM \) and \( AN \), so that the remaining segments \( MB \) and \( NC \) are disjoint and at an angle of one degree. (This is illustrated, for a much larger angle, in the figure.) We think of these line segments as mirrors in that they reflect light perfectly in the usual way.

A light ray in the plane containing \( ABC \) enters quadrilateral \( MBCN \) at the thin end, between \( M \) and \( N \). Then it bounces back and forth until it exits at the thick end, between \( B \) and \( C \). What is the maximum possible number of reflections that it can make?

   Comment: This appeared in a 2001 book of 50 mathematical puzzles, edited by G. Cohen, and we first heard about it from Stan Wagon and Dan Velleman.

5. A race of aliens tours our galaxy looking for planets with edible humanoids. When they visit a planet they take an instantaneous census of all humanoids, and classify each as either edible or inedible. If the number of pairs of edible humanoids is strictly bigger than one-half of the total number of pairs of humanoids, then all of its edibles are promptly hunted and eaten.

   Earth was visited by these aliens sometime in the last century. We (or, at least those of us that were judged to be edible) escaped by the narrowest possible margin. What year did the aliens visit?

   Comment: This problem is due to Steve Silverman.

6. \( N \) people are randomly placed on a line segment. Each person turns to face their closest neighbor (with probability 1, all the distances are distinct, so this is sufficiently well-defined). What is the expected number of “lonely people” who are looking at a neighbor who is not looking at them?
Puzzle solutions follow on next page
Puzzles for Elwyn

Problem 1. Prove that the order of any automorphism of a finite group of order \( n > 1 \) has order less than \( n \).

Comment: This is an extraordinarily difficult problem. You were explicitly asked not to work on it . . . A proof can be found in Horoševskii, M. V., Automorphisms of finite groups. Mat. Sb. (N.S.) 93(135) (1974), 576–587, 630. A more general result asserts that if \( G \) has an automorphism of order larger than \( |G|/2 \) then \( G \) is abelian; this can be found in: Bors, Alexander, Finite groups with an automorphism of large order. J. Group Theory 20 (2017), no. 4, 681–717.

Problem 2. The decimal representation of \( 2^{29} \) has nine distinct digits. Which digit is missing?

Solution: Let \( d \) be the missing digit. Since \( 2^6 \equiv 1 \mod 9 \), we see that \( 2^{29} \equiv 1/2 \equiv 5 \mod 9 \). This is the same thing as the digit-sum (mod 9) of \( 2^{29} \). Since the sum of all 10 digits is \( 45 \equiv 0 \mod 9 \), we have \( 5 + d \equiv 0 \mod 9 \) so that the missing digit is \( d = 4 \).

Problem 3. Each of the twelve faces of a dodecahedron has a light that is also an on/off button. Pushing the light causes all five of the lights on the adjacent faces to switch state (go from on to off, or the reverse). Prove that any of the \( 2^{12} \) positions can be obtained from any other by a suitable sequence of button pushes.

Solution: The Krebs article in the Monthly referenced in the comment in the Emissary gives a general approach to this sort of problem; the key is to showing “full reachability” is to show that a suitable adjacency matrix has full rank. In the dodecahedron case it is easier to note that if the 5 neighbors of a vertex \( v \) are touched, then \( v \) switches state and all other vertices are unchanged.

Problem 4. Find three random variables each uniformly randomly distributed on \([0,1]\) such that their sum is constant. I.e., find a probability distribution on the intersection of \( x+y+z = 3/2 \) with the unit cube in 3-space such that the three “coordinate projections” onto \([0,1]\) are uniform.
Comment/Correction: In the statement of the problem in the *Emissary* the equation of the plane was given as $x + y + z = 1$. In fact, the sum of 3 random variables that are uniform on $[0, 1]$ has mean $3/2$, so if the sum is constant then the sum has to be $3/2$.

Solution: The “trick” solution that was making the rounds among information theorists a few years ago was: Let $X$ be a uniform random variable on $[0, 1]$. If $x$ is in $[0, 1)$ let $x'$ denote the number obtained by writing the ternary expansion of $x$ and incrementing each trit modulo 3. (The reader will surely remember that the ternary expansion of 1 is .2222.) Define random variables $Y$ and $Z$ by giving $Y$ the value $x'$ and $Z$ the value $(x')'$ when $X$ takes the value $x$.

A slightly more prosaic solution is to pick $X$ uniform on $[0, 1]$, let $Y = X + 1/2 \mod 1$, and $Z = 1 - 2X \mod 1$.

Problem 5.  

\(i\). Alice and Bob each have $100 and a biased coin that comes up heads 51% of the time. On a signal, they each start flipping their coins once a minute, betting $1 on the result of the flip. Alice bets on heads; poor Bob bets on tails. As it happens, they both eventually go broke. In that case, who is more likely to have gone broke first?

\(ii\). As above, but this time Alice and Bob bet on the result of the same coin flip (say, flipped by a referee). Again, assume both eventually go broke. Who is more likely to have gone broke first?

Solution: Let $p = 0.51$ and $q = 0.49$.

\(i\). Given that they both go broke, they are equally likely to become broke first.

The probability of Bob going broke is 1, while the probability of Alice going broke is $\left(\frac{q}{p}\right)^{100}$.

Consider a string $B$ of tosses where at the end Bob loses. If we flip it swapping heads and tails, we get a string $A$ where Alice loses. String $B$ contains 100 more heads, while $A$ contains 100 more tails. Such strings are in one-to-one correspondence. Each particular sequence of $2k + 100$ tosses with $k$ tails and $k + 100$ heads ending in bankruptcy for Bob has probability $B_k = q^k p^{k+100}$. Correspondingly, each particular sequence of $2k + 100$ tosses with $k$ heads and $k + 100$ tails ending in bankruptcy for Alice, has probability $A_k = p^k q^{k+100}$. The ratio of these is a constant $\frac{A_k}{B_k} = \left(\frac{q}{p}\right)^{100}$, exactly the apriori probability that Alice goes broke.

That means, they will lose at the same time. Moreover, the distribution of each of them being broken after a given number of tosses is the same, conditioned on the fact that Alice goes broke.

\(ii\). Given that they both go broke, Alice is more likely to go broke first.

Again, swapping heads with tails creates a bijection between cases when Alice goes broke first while Bob is second and vice versa. Since in the first case there are 100 more heads and in the second case there are 100 more tails, each instance of the first case is $\left(\frac{q}{p}\right)^{100}$ more probable than the corresponding instance of the second case. Thus, given that they both go broke, Alice is first with probability $\frac{p^{100}}{p^{100} + q^{100}}$, which is slightly more than 98 percent.
Problem 6. Let $H_n$ be the $n$-dimensional hypercube graph, whose vertices are (labeled by) the $2^n$ binary strings of length $n$, with edges joining vertices whose labels differ in exactly one coordinate. A spider in this graph consists of a central vertex $c$ together with $n$ paths, called legs, that start at $c$. A set of spiders covers $H_n$ if no legs share an edge (though they are allowed to cross at vertices) and every vertex is either a central vertex of some spider, or an endpoint of a spider leg. Prove that there is a collection of spiders that cover with exactly

$$m = \lceil \frac{2^n}{n+1} \rceil.$$ 

Note that this is the minimum possible number, since each spider accounts for $n + 1$ points (its center and edge endpoint of a leg) and there are $2^n$ points.

Comment: This problem is open! Please let us know if you solve it.