Puzzles Column

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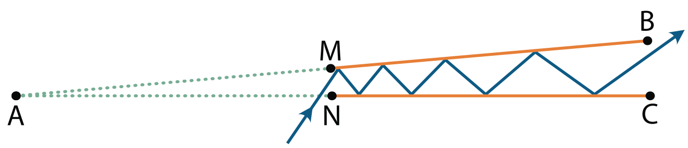
1. Find all possible positive integers a, b, c, and d such that $ab = c + d$ and $cd = a + b$.

2. Let C be the surface of a cube in 3-space. What is the largest n such that there is a regular n-gon P, not lying entirely on one face of C, such that all P's vertices are on C?

3. Here is a different sort of hat problem. There are n hats, each with a different color. These hats are placed on the heads of n sages. All of the sages know all the colors: their own hat and everyone else's hats. A referee then announces the "correct" hat color that should be on the head of each sage.

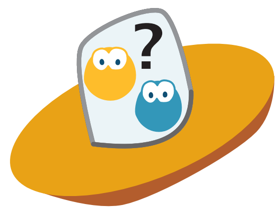
The sages are then allowed "swap" sessions: in one session disjoint pairs of sages are allowed to interchange their hats. Can the sages fully correct their hat colors in two swap sessions?



4. Line segments AB and AC have equal length and have an angle of one degree at A. Let M be the midpoint of AB and N be the midpoint of AC. Erase AM and AN, so that the remaining segments MB and NC are disjoint and at an angle of one degree. (This is illustrated, for a much larger angle, in the figure.) We think of these line segments as mirrors in that they reflect light perfectly in the usual way.

A light ray in the plane containing ABC enters quadrilateral $MBCN$ at the thin end, between M and N. Then it bounces back and forth until it exits at the thick end, between B and C. What is the maximum possible number of reflections that it can make?

Comment: This appeared in a 2001 book of 50 mathematical puzzles, edited by G. Cohen, and we first heard about it from Stan Wagon and Dan Velleman.



5. A race of aliens tours our galaxy looking for planets with edible humanoids. When they visit a planet they take an instantaneous census of all humanoids, and classify each as either edible or inedible. If the number of pairs of edible humanoids is strictly bigger than one-half of the total number of pairs of humanoids, then all of its edibles are promptly hunted and eaten.

Earth was visited by these aliens sometime in the last century. We (or, at least those of us that were judged to be edible) escaped by the narrowest possible margin. What year did the aliens visit?

Comment: This problem is due to Steve Silverman.

6. N people are randomly placed on a line segment. Each person turns to face their closest neighbor (with probability 1, all the distances are distinct, so this is sufficiently well-defined). What is the expected number of "lonely people" who are looking at a neighbor who is not looking at them?

Puzzle solutions follow on next page

Emissary Puzzles Column
Solutions, Spring 2020

May 21, 2020

Problem 1. Find all possible positive integers a, b, c, and d such that $ab = c + d$ and $cd = a + b$.

Solution: First solution. It helps to symmetrize. The two equations imply that

$$(a-1)(b-1) + (c-1)(d-1) = 2, \qquad (a+1)(b+1) = (c+1)(d+1).$$

Assume for the moment that all four variables x are at least 2. Then all of the factors $(x \pm 1)$ in the equations are positive. The only possibility is that the first equation boils down to $1 + 1 = 2$, giving the solution $a = b = c = d = 2$.

Otherwise, up to permutations of the variables, it suffices to consider $a = 1$. The first "symmetric" equation above is then $(c-1)(d-1) = 2$ which means that c and d are 2 and 3 in some order. That means the answer is a suitable permutation of 1, 5, 2 and 3.

Second solution. If $ab = a + b$, then all four variables are equal to 2. Otherwise, either $ab < a + b$ or $cd < c + d$, which means one of the variables is equal to 1. The rest follows as above.

Problem 2. Let C be the surface of a cube in 3-space. What is the largest n such that there is a regular n-gon P, not lying entirely on one face of C, such that all P's vertices are on C.

Solution: The polygon P lies in the plane spanned by any 3 of its vertices, so any P as described has at most 2 points on any face of the cube. Since a cube has 6 faces, certainly $n = 12$ is an upper bound. In fact, this can be achieved! First, note that there is a useful regular hexagon whose vertices are on the cube: Put antipodal vertices of a cube at the north and south pole of a sphere. The equatorial plane, passing through the center of the cube and orthogonal to the long (north-south) axis, intersects 6 edges of the cube. Those 6 intersections form a regular hexagon that is inscribed in the cube. (By symmetry, what else could it be? To visualize this in the real world, you could hold a cube such as a Rubik's cube, by antipodal points, and look.) Cut off just the right amount of each corner of this polygon to give a regular 12-gon with all vertices on the cube.

Problem 3. Here is a different sort of hat problem. There are n hats, each with a different color. These hats are placed on the heads of n sages. All of the sages know all the colors: their own hat and everyone else's hats. A referee then announces the "correct" hat color that should be on the head of each sage.

The sages are then allowed swap sessions: in one such session disjoint pairs of sages are allowed to interchange their hats. Can the sages fully correct their hat colors in two swap sessions?

Solution: Say that a permutation s of the hats is a *swap* if it is the product of disjoint transpositions. (This is the same thing as saying that s has order 2.) In more traditional group theory language, the problem is to show that any permutation of n things can be written as the product of two swaps.

All permutations can be written as a product of cycles, so it suffices to solve the case where s is a cycle. This is easy: write s in cycle form and observe that $s = tu$ where t the "flop" around the midpoint of the cycles, and u is the "flop" around the cycle obtained by removing the first element of s. Examples for $n = 5$ and $n = 6$ are:

$$(12345) = (15)(24) \cdot (25)(34), \qquad (123456) = (16)(25)(34) \cdot (26)(35).$$

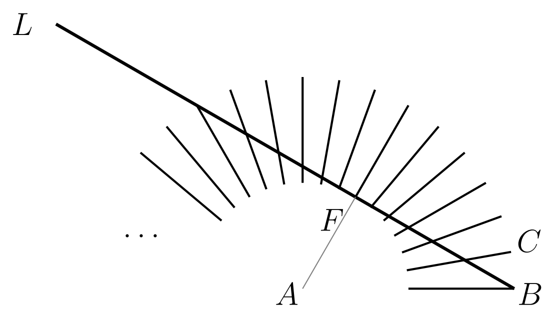
More geometrically, consider a regular n-gon and note that a cyclic permutation of the vertices is just a rotation. Every rotation is a product of two reflections, where the angle between the lines of the reflections is half of the rotation angle. Any reflection of an n-gon is a swap permutation on its vertices.

Problem 4. Line segments AB and AC have equal length and have an angle of 1 degree at A. Let M be the midpoint of AB and N be the midpoint of AC. Erase AM and AN, so that the remaining segments MB and NC are disjoint and at an angle of 1 degree. We think of these line segments as mirrors in that they reflect light perfectly in the usual way.

A light ray in the plane containing ABC enters quadrilateral $MBCN$ at the thin end, between M and N. Then it bounces back and forth until it exits at the thick end, between B and C. What is the maximum possible number of reflections that it can make?

Solution: As often the case for problems with mirrors, it is useful to reflect the whole picture with respect to the mirrors, as many times as is necessary, so that the light paths correspond to straight lines. For the sake of the description below it is convenient to consider rays that enter from the fat end of the "megaphone"; note that the maximum possible number of reflections is the same (and it gives a clear demonstration of the curious fact that if a light ray enters the fat end at too shallow an angle, then the light will end up returning out of the fat end).

For clarity, in the diagram below the angle of the megaphone is 10 degrees rather than 1 degree. The megaphone is reflected a number of times in the diagram. The path of a light ray ray entering the megaphone at a point E between B and C (without loss of generality, with its first reflection is on the line AC) corresponds (under the reflections of the megaphone) to a straight line through E. The line L corresponds to the case $E = B$, exiting the megaphone at the point (corresponding under multiple reflections to) F, where ABF is a 30-60-90 right triangle. Let L' be the analogous line starting at C that is tangent to the inner circle of radius 1 at a point F'. If the actual light ray L_E starting at an arbitrary E in between B and C does not enter the inner circle then this light ray comes out the wrong end of the megaphone. If L_E enters the inner circle between F and F' then is has reflected 60 times. If L_E enters the inner circle below F then it has reflected fewer than 60 times. In other words, the maximum possible number of reflections is 60.



Problem 5. A race of aliens tours our galaxy looking for planets with edible humanoids. When they visit a planet they take an instantaneous census of all humanoids, and classify each as either edible or inedible. If the number of pairs of edible humanoids is strictly bigger than one-half of the total number of pairs of humanoids, then all of the planet's edibles are promptly hunted and eaten. Earth was visited by these aliens sometime in the last century. We (or, at least those of us that were judged to be edible) escaped by the narrowest possible margin.

What year did the aliens visit?

Solution: Suppose there were h people altogether on the planet at the exact time of their visit, and e of those humans were edible. To fail by the narrowest of possible margins means that the number of pairs of humans was *exactly* twice the number of pairs of edible humans, i.e.,

$$2 \cdot \frac{e(e-1)}{2} = \frac{h(h-1)}{2}.$$

Multiplying the equation $2(e^2 - e) = h^2 - h$ by 4 and completing the square gives

$$2y^2 = x^2 + 1, \quad \text{where } \ x = 2h - 1 \text{ and } y = 2e - 1.$$

Lo and behold, one of the simplest Pell's equations, $x^2 - 2y^2 = -1$, has emerged. All solutions have the form $x + y\sqrt{2} = (1 + \sqrt{2})^n$ for n odd. Chasing through all of the algebra, and using $x \simeq \sqrt{2}y$, gives $h \simeq (1+\sqrt{2})^n/4$. The only value of h of this form that occurred in the 20$^{\text{th}}$ century is $h = 5{,}406{,}093{,}004$, and most world population tabulations put this as having happened late in 1991 (a few say that it happened in early 1992).

Problem 6. N people are randomly placed on a line segment. Each person turns to face their closest neighbor (with probability 1, all the distances are distinct, so this is sufficiently well-defined). What is the expected number of "lonely people" who are looking at a neighbor who is not looking at them?

Solution: We are only interested in the relative lengths of the distances between neighbors. Thus the distances can be viewed as a permutation of $N-1$ elements. The pair of neighbors looking at each other corresponds to a valley in the permutation: that is the distance between them is smaller than the distances from them to their other neighbors. The probability that the first or last number in the permutation is a valley is $\frac{1}{2}$. The probability for other numbers in the permutation to be a valley is $\frac{1}{3}$. By linearity or expectation, the expected number of valleys is

$$\frac{1}{2} + \frac{1}{2} + \frac{N-3}{3} = \frac{N}{3}.$$

Each valley corresponds to two people looking at each other. Thus the expected number of lonely people is

$$N - 2 \cdot \frac{N}{3} = \frac{N}{3}.$$