Adventures in constructive Galois theory

Understanding Galois extensions of fields is a central problem in algebra, with a number of open questions, accessible at a number of levels. At the core, Galois theory is an attempt to understand the arithmetic of fields, by studying the types of equations one can set up over a given field, and the structure and symmetries of their sets of solutions.

In this project we will explore some topics along the edges of "explicit inverse Galois theory," which tries to understand which groups arise as Galois groups for a given field, and how. Our goal will be to take constructive approaches to work in a less explored direction with these Galois extensions to understand richer algebraic structures and properties that collections of Galois extensions exhibit as a whole, in particular looking for reflections of the kinds of structure one seems in Kummer theory.

As a simple example, one could consider quadratic extensions of a field F, which are those whose Galois group is C_2. If the characteristic of F is not 2, these are all obtained by adjoining a root of a polynomial of the form $x^2 - t$. The collection of all such field extensions, up to isomorphism, themselves form a group, isomorophc to $F^*/(F^*)^2$, where field extensions with parameters t_1, t_2 "add together" to get the field extension with parameter $t_1 t_2$.

On the other hand, field extensions of F whose Galois group is C_3, where the characteristic of F is not 3. It turns out that these can always be by adjoining a root of a polynomial in the simple, but somewhat less intuitive form $x^3 - tx^2 + (t - 3)x + 1$. Again, it turns out that these again form a group, but the structure of this group, while carrying imporant information about F, is not so easily expressed in the parameter t. In part, we will explore other ways of describing these fields in order to make the group structure more transparent and intuitive.