Emissary Puzzles Column
Solutions, Fall 2020

Problem 1. We are given eight unit cubes. The faces of each cube are colored either blue or red. One third of all of those faces are red. We build a $2 \times 2 \times 2$ cube out of these cubes so that exactly one third of the unit cube's faces visible on the faces of the larger cube are red. Prove that you can use these cubes to build a $2 \times 2 \times 2$ cube whose faces are entirely red.

Solution: We are chagrined to have to say this, but there is a bad typo in the statement of the problem. The sentence "One third of all of those faces are **red**" should have been "One third of all of those faces are **blue**." (Indeed, if only 1/3 of the 48 faces are red, i.e., 12 of them, then it is impossible for the 24 exterior faces of a $2 \times 2 \times 2$ stack to all be red.)

Assume the corrected statement. If all 16 blue faces are on the exterior of a $2 \times 2 \times 2$ stack, then all interior faces have to be red, and a stacking with only red faces can be obtained by "reversing" each cube by flipping it by putting its unique interior corner outside on the corresponding corner of the $2 \times 2 \times 2$ stack.

Problem 2. Let A be an n by n matrix, and B the 90-degree clockwise rotation of that matrix (so that, for example, $B_{1,n} = A_{1,1}$). Find, and prove, a relationship between $\det(A)$ and $\det(B)$.

Solution: A 90-degree rotation can be written as the composition of a taking the transpose, and reflecting about a central vertical axis. Taking the transpose leaves the determinant unchanged. A reflection about a central axis swaps $(n-1)/2$ pairs of columns if n is odd, and $n/2$ columns if n is even. Each swap changes the sign of the determinant. The result could be written

$$\det(B) = (-1)^{\lfloor n/2 \rfloor} \det(A).$$

Another way to express the result is to say that rotating a matrix by 90 degrees leaves the determinant unchanged, up to sign, and it changes the sign of the determinant if and only if n is congruent to 2 or 3 modulo 4.

Problem 3. Alice tosses 99 fair coins and Bob tosses 100. What is the probability that Bob gets more heads than Alice?

Solution: The events "Bob throws more heads than Alice," and "Bob throws more tails than Alice" are mutually exclusive and complementary. Moreover, one of them has to be true (Bob has more throws than Alice). By symmetry (exchanging "heads" and "tails") their probabilities are equal, so both are equal to 1/2.

Problem 4. The Board of Directors of the Acme Acute Angles Company has grown too large. It has 50 members, and they have agreed to the following reduction protocol:

The Board will vote on whether or not to reduce its size. If a majority vote "yes," the newest member is ejected from the Board. If that happens a new yes/no vote is taken, and this continues until half or more of the surviving members vote "no," at which point the protocol ends, and the Board is fixed.

Suppose that each member places the highest priority on personally remaining on the Board, but, given that, agrees that the smaller the Board, the better.

To what size will this protocol reduce the Board?

Solution: Suppose, more generally, that the Board has n members. Working carefully through a few cases of small n strongly suggests that the Board will always vote itself down to the largest possible power of 2, so that in the stated case $n = 50$ the Board will end up with 32 members.

This is easy to prove by induction. The initial cases are trivial. If the Board has n members with $2^k < n < 2^{k+1}$, then the oldest-serving 2^k members see that the Board will be reduced, but they will be safe, so a majority vote will expel the newest member. If $n = 2^{k+1}$ then the newest $n = 2^k$ members see (by induction) that they will ultimately not be on the Board if they vote to decrease the Board, so they vote "no," the vote is a tie, and the process stops.

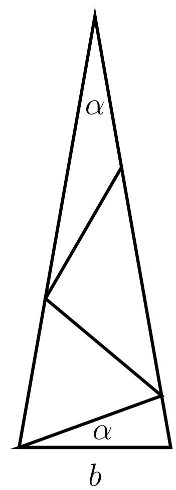
Problem 5. An isosceles triangle is divided into 4 smaller isosceles triangles. Is it true that you can always find a pair of congruent triangles among them?

Solution: Any attempt with two of the triangles being similar seems to fail. That suggests something like the diagram below of 4 isosceles triangles. Each of the 4 isosceles triangles has two sides of whose length is the length, b, of the base of the large triangle.

The triangle at the bottom is similar to the entire stack of all of the 4 triangles, so the top angle α is equal to the indicated angle in the bottom triangle. The remaining angles can all be filled in from the top down or bottom up, leading to a linear equation in α whose solution is

$$\alpha = \frac{\pi}{9} = 20^\circ.$$

The second triangle up is equilateral, and the duplicated angles in the other triangles are 20, 40, and 80 degrees respectivelly.



Problem 6. Eight out of sixteen coins are heavier than the rest, and weigh 11 grams each. The other eight coins weigh 10 grams each. We do not know which coin is which, but one coin is conspicuously marked as a "Special" coin. Can you figure out whether the Special coin is heavier or lighter using a balance scale at most three times?

Solution: Let S denote the Special coin. The remaining coins are labeled according to their role in the first balance test: Chose 6 coins $L_1, \ldots, L_6$ to go on the left pan with S, and 7 other coins $R_1, \ldots, R_7$ to go on the right pan. This leaves two coins U_1, U_2 that are unused in the first balance.

Suppose that the first balance is right heavy. (Note that then most 3 of the coins on the left could be heavy). Now balance S against L_1. If that is unbalanced, you are done. If it balances, use your third balance to be S, L_1 versus L_2, L_3. Again, if that is unbalanced, you are done. However, if it balances, then by the comment all four coins are light!

Now suppose that the first balance is in fact an exact balance. (Note that this means that the two unused coins U_1, U_2 have the same weight.) Now balance S and L_1 against U_1 and U_2. If that is unbalanced, then weighing S against L_1 determines the answer. If the second weighing is a balance, then compare S, L_1, U_1, U_2 against L_2, L_3, L_4, L_5. If this is unbalanced, the question has been answered, because you know that all 4 coins on the left have the same weight. And this has to happen! Indeed, if this third weighing was in balance, it would be impossible for the first weighing to balance.

This is essentially the "official" solution for the Moscow Math Olympiad, where this appeared. There are others, and Peter Winkler (who also recently used this in in his "Mind Benders for the Quarantined" emailing) remarks that no non-adaptive solution (where all weighings are announced in advanced) seems to be known.

Note that there is an easy non-adaptive solution if 4 weighings are allowed. If S and 15 other coins $C_1, \ldots, C_{15}$ are given, then weigh S vs. C_1, those two against C_2 and C_3, then those four against C_4, C_5, C_6, C_7, and finally those 8 against the rest. Winkler also notes that you will actually have to do the fourth weighing only if you are incredibly unlucky in choosing the labels for thecoins, and end up with $C_1, \ldots, C_7$ all have the same weight as S.