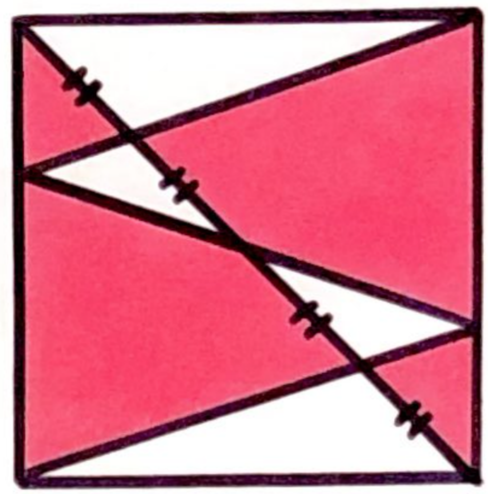
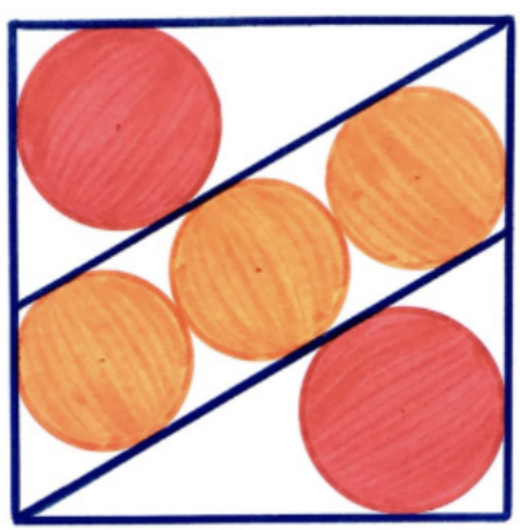
**Emissary** Puzzles Column, Solutions, Spring 2021

**Problem 1.** What fraction of the square is shaded?



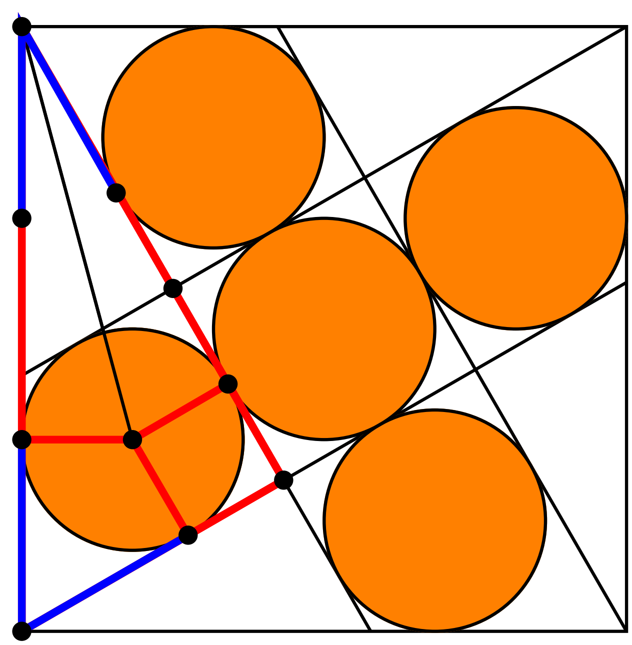
**Solution:** The two small red triangles have the same area as their nearby small white triangles. If those colors are reversed, one checks that then the red area can be deconm-posed into 4 triangles similar to the 2 white triangles, so the red area is twice the area of the white area, and two-thirds of the square is red.

**Problem 2.** "Five Circles in a Square." The total red area is 24. What is the total orange area?



**Solution:** The angle of the four slanted lines is determined completely by the fact that they bound three circles as indicated. Since the middle circle is central, we temporarily erase the red circles and add two orange circles to get the symmetrical diagram shown below. It is not too hard to do some algebra to see that all of the right triangles formed in the diagram are 30/60/90 triangles. This algebrauc exertion is not something that Catriona Agg would be happy with. In a discussion of this problem (on Twitter) we found a diagram which shows this fact "purely" geometrically using the added red and blue lines. Namely, the congruence of the two thin triangles (each with a vertex in the upper left corner) shows that the hypotenuse side of the large triangle has 2 red and 2 purple segments, whereas the short side as 1 of each; this shows that the triangle is indeed half of an equilateral triangle, as claimed.

The ratio of the areas of incircles of similar triangles is equal to the ratio of their areas. Comparing with the original figure, we see that the triangle with yellow incircle is obtained from the triangle with red incircle by multiplying all of its sides by $\sqrt{3}/2$, so the ratio of the areas is $3/4$. Since their are 2 red triangles and 3 yellow ones, the yellow area is 27.



**Problem 3.** Factor 2021, in your head!

**Solution:** There are no obvious factors, so the next thing to try is to express 2021 as a difference of squares. Aha! Since $45^2 = 2025$, we see that $2021 = 45^2 - 2^2 = 43 \cdot 47$.

**Problem 4.** Does there exist a 19-gon inscribed in a circle, with no two sides of equal length, whose angles are all an integer number of degrees?

**Solution:** Assume that a 19-gon has sides $s_i$, where $1 \leq i \leq 19$, let $\alpha_i$ be the corresponding central angle of the circle. The isosceles triangle formed by the center of the circle and two successive vertices has angles $\alpha_i$, $90 - \alpha_i/2$, and $90 - \alpha_i/2$. Since the sides are distinct, all $\alpha_i$ are distinct. Moreover, the angle between the sides $s_i$ and $s_{i+1}$ is $(180 - \alpha_i - \alpha_{i+1})/2$ (where by convention $\alpha_{20} = \alpha_1$). The sum of the $\alpha_i$ is 360, and each of the pair sums $\alpha_1 + \alpha_2$, $\alpha_3 + \alpha_4$, $\ldots$, $\alpha_{17} + \alpha + 18$ is even; thus $\alpha_{19}$ is even and (by symmetry) all $\alpha_i$ are even. The sum of the angles of a 19-gon is $17 \cdot 360$ (which is equivalent to saying that the sum of the exterior angles $\alpha_i + \alpha_{i+1}$ is 360). However, the sum of the first 19 even integers $19 \cdot 20 = 380$, which is a contradiction!

**Problem 5.** Define a sequence $x_i$ by setting $x_i = 1$ for $1 \leq i \leq 2021$, and

$$x_{i+1} = x_i + x_{i-2020}, \quad i \geq 2021.$$

Prove that the sequence has 2020 consecutive terms that are divisible by 2021.

**Solution:** Let $n = 2021$. Note that the recursion can be run backward by rewriting it as

$$x_{k-n} = x_k - x_{k-1}$$

to define $x_i$ for all integers $i$. One finds that $x_k = 0$ for $-(n-1) < k \leq 0$, i.e., the sequence has $n-1$ consecutive 0s. Since for any "modulus" $M > 0$ there are only finitely many $(n-1)$-tuples modulo $M$, and the fact that any consecutive run of $n-1$ values determines the sequence going in both directions, it follows that the sequence is ultimately periodic modulo $M$. So for any $M$ (and in particular, $M = n$) there are $n-1$ values of consecutive $x_i$ (all with $i > 0$) that are divisible by $M = n$.

**Problem 6.** Alice and Bob know that during the round of 16 teams in the NCAA basketball tournament they will be in different cities in a situation where they can communicate only via a very expensive two-way communication channel that transmits single bits at a time. They know that Alice will know which two teams are playing, but not who won, and Bob will know who won. Find a communication protocol that enables Bob to communicate to Alice who won by exchanging exactly 3 bits.

**Solution:** There is an obvious 4-bit solution: Beforehand (when they are in the same city) Alice and Bob agree on an encoding of the 16 teams as 4-bit strings, and Bob sends the 4 bits corresponding to the winning team.

Here is a protocol that uses 3 bits. Beforehand, they also agree on a 2-bit encoding of any of the 4 positions in the names of the teams. Alice will send the 2-bit name of a position in which the (names of) the two teams playing differ. (There will of course be at least one such position, but there might be more.) Bob will look at the position in the name of the winning team, and send that bit (0 or 1) to Alice, who will then know who won.

**Comment:** Alon Orlitsky's original IEEE paper on (a vast generalization of) this appeared in 1989, entitled *Worst-case Interactive Communication I: Two Messages are Almost Optimal*. This appeared in Peter Winkler's recent book, and he used it recently in his *Mind-Benders for the Quarantined* weekly problem.

**Problem 7.** A combination lock has three dials, each of which can be set in any one of four positions. The lock opens if two dials are correct. What is the smallest number of positions that are guaranteed to open the lock if they are all tried.

**Solution:** This can be done with the 8 settings

$$(1,1,2),(1,2,1),(2,1,1),(2,2,2),(4,4,3),(4,3,4),(3,4,4),(3,3,3).$$

Indeed, any 3-tuple of elements of $\{1,2,3,4\}$ will either have 2 elements in $\{1,2\}$ or 2 elements in $\{3,4\}$, and the first 4 settings cover the first case, and the last 4 cover the latter case.

Second, here is an argument that 7 settings will not suffice. Suppose in fact that there was a set $S$ of 7 3-vectors with components in $[4] := \{1,2,3,4\}$ that cover $V = [4]^3$ in the (Hamming distance) sense that any element of $V$ can be changed in one component to get an element of $S$. Some number occurs at most once in the first component of those 7 vectors. Say that 1 occurs only once. If the vector is $(1,a,b)$ then there are 9 vectors of the form $(1,x,y)$ where $x$ is not equal to $a$ and $y$ is not equal to $b$. Those vectors have to be covered with the remaining 6 vectors, none of which has a 1 in the first component; this is impossible. (The case where none of the 7 vectors has a 1 in the first component is even easier.)

**Problem 8.** Show that every positive integer can be written as a sum of numbers of the form $2^a 3^b$ in such a way that no summand is a divisor of some other summand.

**Solution:** Let $n$ be the smallest integer for which such a decomposition is not yet known. We can assume that $n > 1$ and $n$ is not divisible by 2 or 3. If $3^k$ is the largest power of 3 that is less than $n$ then $n - 3^k$ is even so that

$$n = 3^k + 2x.$$

for some integer $x$. Substituting the known expression for $x$ gives one for $n$. Note: the date of the attribution is wrong: this was the first problem on the 2005 Putnam.

**Problem 9.** Tetrahedral hats have become all the rage, and a group of $n > 1$ people have hats on their heads that are shaped like regular tetrahedrons: 4 vertices, 6 edges, 4 faces. (Moreover, it should be obvious that the correct way to wear a tetrahedron is with one of the triangular faces on the head with one vertex at the peak, and another facing forward, so that vertices can be unambiguously labeled as front, left, right, and top on everyone's head.)

A referee goes around an puts a visible mark on one edge of each of the $n$ hats. At a specified signal the players simultaneously name a vertex on their hat. The players collectively win if and only if *all* of the players name a vertex on the marked edge on their hat. All the usual rules apply: everyone can see all hats but their own, the players have a strategy session the night before, no communication (covert or otherwise) is allowed during the game, and their statements are exactly simultaneous.

What strategy gives the players their highest chance of winning?

**Comment:** The problem is still unsolved. It comes up briefly, on p. 16 in the paper "On the Notorious Levine Hat Puzzle" that will appear in the online journal *Integers* (and can be found on the arXiv).