Higher Operators on Resolutions over Complete Intersections

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Outline

Construction of the CI Operators

• $f_1, \ldots, f_c \in S$ a regular sequence in a regular local ring (or standard graded polynomial ring). $k = S/\mathfrak{m}$ the residue field of S
• $R = S/(f_1, \ldots, f_c)$; and M a finitely generated R-module
• $\mathbb{F} : \cdots \xrightarrow{d} F_1 \xrightarrow{d} F_0$ a minimal R-free resolution of M
•$\widetilde{\mathbb{F}} : \cdots \xrightarrow{\tilde{d}} \widetilde{F}_1 \xrightarrow{\tilde{d}} \widetilde{F}_0$ a lifting of d to a sequence of maps $\tilde{d}$ of free modules over S (*not a complex!*)

Write $\tilde{d}\tilde{d} = \sum_i f_i \tilde{t}_i$ (not unique). Set $t_i := R \otimes \tilde{t}_i : F_i \to F_{i-2}$.

Theorem:

[E, 1980] *The t_i are maps of complexes, $F \to F[2]$. They are homotopy-unique and homotopy-commutative. Thus they define a module structure on $E = \mathrm{Ext}_R(M, k) = \sum_{p \geq 0} \mathrm{Ext}^p_R(M, k)$ over $T := k[t_1, \ldots, t_c]$. The module E is finitely generated over T of Krull dimension $\leq c$.*

Codimension 0

There are no CI operators in this case; but there is a pattern:
$c = 0,\ R = S$, regular local; minimal resolutions are finite, and
$Ext_R(M, k)$ is a T-module of finite length.
Assymptically, $\mathrm{Ext} \equiv 0$.

The finite part is still mysterious! For example:

Conjecture

[Buchsbaum-E-Horrocks]: *Length is* $\geq 2^{\mathrm{codim} M}$

Codimension 1

$c = 1,\ R = S/(f)$, hypersurface. A high truncation of any $T = k[t]$-module is free. Thus:

Theorem:[E, 1980]*If M is a high R-syzygy (any Cohen-Macaulay module) then:*

- $t = t_1 : \mathbb{F}[-2] \to \mathbb{F}$ *is an isomorphism;* $\mathbb{F}$ *is periodic of period 2; all* F_i *have the same rank.*
- $\widetilde{d}_{i+1}\widetilde{d}_i = f \cdot Id$.

Definition: a **matrix factorization** of a nonzerdivisor f is a pair of maps of free S-modules $F \xrightarrow{\widetilde{d}} G \xrightarrow{\widetilde{d}'} F$ such that F, G have the same rank and $\widetilde{d}\widetilde{d}' = f \cdot Id$.

(Irena Peeva spoke about our work on a generalization: higher matrix factorizations. Not my subject today.)

Two Modules

Because the t_i have degree 2, $\mathrm{Ext}_R(M, k)$ is the direct sum of *two* $k[t_1, \ldots, t_c]$-modules: $\mathrm{Ext}^{even}(M, k) := \sum_p \mathrm{Ext}_R^{2p}(M, k)$ and $\mathrm{Ext}^{odd}(M, k) := \sum_p \mathrm{Ext}_R^{2p+1}(M, k)$.

What's the relationship between $\mathrm{Ext}^{even}(M, k)$ *and* $\mathrm{Ext}^{odd}(M, k)$*??*

Avramov, Gasharov and Peeva proved that the two modules have same Krull dimension, same multiplicity; this generalizes the fact that for $c = 1$, if M is a Cohen-Macaulay module (and not free) then in the minimal resolution

$$\mathbb{F} : \cdots \to F_1 \to F_0$$

of M, all the F_i have the same rank.

They also proved that, in any codimension, for $i \gg 0$ the ranks of the F_i are either constant or strictly increasing.

Two Modules, $c = 2$

What about Krull dim$\mathrm{Ext}(M, k) = 2$? Example:
$R = k[x, y]/(x^3, y^3)$, $M = R/(x, y)^2$. The minimal free resolution is

$$\mathbb{F} : \cdots \to R^7 \to R^6 \to R^4 \to R^3 \to R.$$

so

$$\mathrm{Ext}^{even} = T \oplus T^2(-2), \qquad \mathrm{Ext}^{odd} = T^3(-1).$$

In general (since Betti numbers will be strictly increasing!) the free part will be, up to truncation,

$$\mathrm{Ext}^{even} = \oplus_1^m T(a_i), \qquad \mathrm{Ext}^{odd} = \oplus_1^m T(b_i)$$

where $\sum_1^m a_i < \sum_1^m b_i < m + \sum_1^m a_i$.

Higher CI operators

We can put all the $\tilde{t}_i$ together as a map

$$\tilde{t} : \mathbb{F} \to \mathbb{F}[-2] \otimes S^c.$$

Let

$$\mathbb{K} : 0 \to \wedge^c S^c \xrightarrow{\delta} \cdots \xrightarrow{\delta} \wedge^2 S^c \xrightarrow{\delta} \wedge^1 S^c \xrightarrow{\delta} S$$

be the Koszul complex of $f_1, \ldots, f_c$, and consider the array of maps $\widetilde{\mathbb{F}} \otimes_S \mathbb{K}$. The equation that defines $\tilde{t} : \mathbb{F} \to \mathbb{F}[-2] \otimes S^c$ is $(1 \otimes \delta)(\tilde{t}) = (\tilde{d} \otimes 1)^2$. If we set

$$s_0 = 1 \otimes \delta \quad s_1 = \tilde{d} \quad s_2 = t$$

this becomes

$$s_0 s_2 = s_1^2 : \mathbb{F} \otimes K_0 \to \mathbb{F} \otimes K_1$$

Moreover, taking into account the fact that the homological degree -1 term of $\mathbb{K}$ is 0, we may harmlessly add the term $s_2 s_0$, which is 0 on $\mathbb{F} \otimes K_1$.

Higher CI operators—continued

We had:

$$s_0 s_2 = s_1^2 : \mathbb{F} \otimes K_0 \to \mathbb{F} \otimes K_1$$

Taking into account the fact that the homological degree -1 term of $\mathbb{K}$ is 0, we may harmlessly add the term $s_2 s_0$, which is 0 on $\mathbb{F} \otimes K_1$, and write this as $s_0 s_2 + s_2 s_0 = s_1^2$

Theorem

There are operators $s_i : \widetilde{\mathbb{F}} \otimes \mathbb{K} \to \mathbb{F}[i] \otimes \mathbb{K}[-i+1]$ *such that* $s_0 = 1 \otimes \delta \quad s_1 = \tilde{d} \quad$ *and, for all n,*

$$\sum_{i+j=n} s_i s_j = 0.$$

Construction

With $R = S/(f_1, \dots, f_c)$ and an R-module M as before, let

$$\mathbb{G} : \cdots \xrightarrow{u_2} G_1 \xrightarrow{u_1} G_0$$

be a free resolution of M as an S-module. Since M is annihilated by $f_1, \dots, f_c$ there is an isomorphism $R \otimes_S M \to M$, and this lifts to a map of complexes

$$\mathbb{K} \otimes_S \mathbb{G} \to \mathbb{G}$$

The Homotopy uniqueness of comparison maps guarantees that, modulo homotopy, this gives G the structure of a differential graded $\wedge S^n$-module. In particular,

$$Tor^S(M, k) \text{is a module over } \wedge R^c$$

An example with $c = 3$:

$$S = k[x, y, z] \quad R = S/(x^3, y^3, z^3), \quad M = R/(x, y, z)^2$$

Note that M is a Cohen-Macaulay module. The minimal free resolution of M as S-module is

$$0 \to S^3 \to S^8 \to S^6 \to S$$

On the next slide are the betti tables of the $\wedge k^3$-free resolutions of $\mathrm{Tor}^S(M_i, k)$ where M_i is the i-th syzygy of $M = M_0$ and $i = 0, 1, 2$

	0	1	2	3	4	5	6	7	8	9	10
total:	18	54	108	180	270	378	504	648	810	990	1188
0:	1	3	6	10	15	21	28	36	45	55	66
1:	6	18	36	60	90	126	168	216	270	330	396
2:	8	24	48	80	120	168	224	288	360	440	528
3:	3	9	18	30	45	63	84	108	135	165	198

	0	1	2	3	4	5	6	7	8	9	10
total:	20	57	112	185	276	385	512	657	820	1001	1200
0:	6	18	36	60	90	126	168	216	270	330	396
1:	11	30	58	95	141	196	260	333	415	506	606
2:	3	9	18	30	45	63	84	108	135	165	198

	0	1	2	3	4	5	6	7	8	9	10
total:	32	75	136	215	312	427	560	711	880	1067	1272
0:	11	30	58	95	141	196	260	333	415	506	606
1:	21	45	78	120	171	231	300	378	465	561	666

The case of a high syzygy

Theorem (E-Peeva-Schreyer)

If M is a high syzygy over the complete intersection $R = S/(f_1, \ldots, f_c)$, then $\mathrm{Tor}^S(M, k)$ is generated by Tor_0 and Tor_1. Moreover, the resolution of $\mathrm{Tor}^S(M, k)$ as a $\wedge k^c$-module has only two linear strands. The Koszul duals of these strands are $\mathrm{Ext}_R^{even}(M, k)$ and $\mathrm{Ext}_R^{odd}(M, k)$. The maps between the two strands of the resolution are given by the higher CI operator s_3.