Tate Resolutions on Products of Projective Spaces

David Eisenbud, Daniel Erman and Frank-Olaf Schreyer

University of California Berkeley, University of Wisconsin and Universität des Saarlandes

Baltimore, January 16, 2014

Notation

We work on projective space $\mathbb{P}^n := \mathbb{P}(W)$ with homogeneous coordinate ring $S := Sym(W)$.

The Tate resolution is a complex of free modules over $E := wedgeW^*$ associated to a coherent sheaf $\mathcal{F}$ on $\mathbb{P}(W)$.

For convenience we set $\omega_E = E(-n-1)$; this is the free E-module of rank 1 with generator in degree $n+1$ and socle in degree 0. (Note: the elements of W^* have degree -1.)

Review: Tate Resolution of a coherent sheaf

Theorem (Eisenbud-Fløystad-Schreyer)

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. There is a doubly infinite exact complex of free E-modules with terms

$$T^d = \bigoplus_{0 \leq i \leq n} H^i \mathcal{F}(d-i) \otimes_k \omega_E(i-d).$$

such that for $d >> 0$ the differential is

$$\partial : T^d \ni \sigma \otimes \epsilon \mapsto \sum_j x_j \sigma \otimes e_j \epsilon \in T^{d+1}$$

where $\{x_i\}, \{e_i\}$ are dual bases of W, W^*

Definition:

$$\mathbf{T} : \cdots \to T^d \to T^{d+1} \to \cdots$$

is the Tate Resolution of $\mathcal{F}$.

Usefulness of the Tate Resolution

Applications (many authors) include computation (for example in the sense of Macaulay2) of:

- Beilinson monads: (apply the functor $\omega_E(i) \mapsto \wedge^i U$, where U is the universal sub-bundle on $\mathbb{P}^n$, to the Tate Resolution)
- Chow forms, resultants
- Boij-Söderberg theory
- direct image complexes (local or affine case)
- higher cohomology operations

What makes it work?

For $d \gg 0$ we have $H^i(\mathcal{F}(d)) = 0$ for $i > 0$, so

$$T^d = H^0\mathcal{F}(d) \otimes \omega_E(-d)$$

and in this range giving the complex $T^{\geq d}$ is equivalent to giving the finitely generated $S = Sym(W)$-module

$$\oplus_{d' \geq d} H^0\mathcal{F}(d').$$

The rest of the Tate resolution can thus be computed by computing the free resolution of the cokernel of an easy-to-compute map of finitely generated free E-modules,

$$H^0\mathcal{F}(d) \otimes \omega_E(-d) \to H^0\mathcal{F}(d) \otimes \omega_E(-d)$$

$$\sigma \otimes \epsilon \mapsto \sum_j x_j\sigma \otimes e_j\epsilon.$$

What about products $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$?

Why not...just replace all indices with multi-indices??

Give $W, S = Sym(W), E = \wedge(W^*)$ all $\mathbb{Z}^t$ gradings, set $n = (n_1, \ldots, n_t)$.

Question: Given a coherent sheaf on $\mathbb{P}(W)$, is there a doubly infinite multigraded exact complex of free E-modules with terms

$$T^d = \sum_{|c-i|=d} H^{|i|}\mathcal{F}(c-i) \otimes \omega_E(i-c)$$

with, for $d \gg 0$, differential $\partial : \sigma \otimes \epsilon \mapsto \sum_j x_j\sigma \otimes e_j\epsilon$

(now $c, i \in \mathbb{Z}^t$ and $0 \leq i \leq n$ termwise)?

There are problems...

- Even for $d \gg 0$ and the sheaf $\mathcal{F} = \mathcal{O}_{\mathbb{P}}$ on $\mathbb{P} := \mathbb{P}^1 \times \mathbb{P}^1$, the sum
$$T^d = \sum_{|c-i|=d} H^{|i|}\mathcal{F}(c-i) \otimes \omega_E(i-c)$$
has infinitely many nonzero terms
$$H^1\mathcal{O}(c_1, c_2) = H^0\mathcal{O}_{\mathbb{P}^1}(c_1) \otimes H^1\mathcal{O}_{\mathbb{P}^1}(c_2) \neq 0$$
with $c_1 \gg 0$ and $c_2 \ll 0$.
- Even in the case of a single factor $\mathbb{P}^n$, when there are terms $H^i\mathcal{F}(d) \neq 0$ with $i > 0$ the differential is not ∂—we don't know a formula.
- So... **NO! There is no such complex** with differential as above for $d \gg 0$.

The Good News

Theorem (Eisenbud-Erman-Schreyer)

Given a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^{n_1} \times \mathbb{P}^{n_t}$ there **IS** *a doubly infinite exact complex of free (infinitely generated) E-modules T with terms*

$$T^d = \sum_{|c-i|=d} H^{|i|}\mathcal{F}(c-i) \otimes \omega_E(i-c).;$$

Although we don't know a formula for any differential $T^d \to T^{d+1}$, T has so many exact subcomplexes that we can compute the restriction of T, with its differential, in any bounded "box" $b_{lower} \leq c \leq b_{upper}$, allowing applications as in the one-factor case.

A first quadrant complex

For simplicity, restrict to the case of two factors $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$.

Choose $d \gg 0$. Consider the total complex $\mathbf{T}_+$ of the "first quadrant double complex" with terms

$$H^0\mathcal{F}(c_1, c_2) \otimes \omega_E(-c_1, -c_2)$$

with $c_1 \geq b$ and $c_2 \geq b$, where the differential ∂_{hor} is constructed from the variables of the first factor and the differential ∂_{vert} is constructed from the variables of the second factor. The first map is

Δ:

$$H^0\mathcal{F}(b, b) \otimes \omega_E(-b, -b) \to H^0\mathcal{F}(b+1, b) \otimes \omega_E(-b-1, -b)$$
$$\oplus H^0\mathcal{F}(b, b+1) \otimes \omega_E(-b, -b-1) \to \cdots$$

.

A third-quadrant complex

The kernel of the first component of Δ,

$$\partial_{hor} : H^0\mathcal{F}(b,b)\otimes\omega_E(-b,-b) \to H^0\mathcal{F}(b+1,b)\otimes\omega_E(-b-1,-b)$$

is

$$H^0\mathcal{F}(b-1,b)\otimes\omega_E(-b+1,-b) \to H^0\mathcal{F}(b,b)\otimes\omega_E(-b,-b)$$

and similarly for the second component ∂_{vert} of Δ. So the kernel of Δ "should" be the "common part"; that is, the image of

$$H^0\mathcal{F}(b-1,b-1)\otimes\omega_E(-b+1,-b+1)$$

via the "corner composition"

$$\partial_{\ulcorner c} := \partial_{hor}\partial_{vert} = \pm\partial_{vert}\partial_{hor}.$$

The Corner complex

This is true!

Theorem

The free resolution of $\ker \partial_{\Rsh c}$ *is the "third quadrant" subcomplex* $\mathbf{T}_-$ *of the Tate resolution with terms*

$$T^d_{\Rsh c} = \sum_{|c-i|=d, \quad c_j<b} H^{|i|}\mathcal{F}(c-i) \otimes \omega_E(i-c).$$

We write $\mathbf{T}_{\Rsh c}$ for the result of splicing $\mathbf{T}_-$ with $\mathbf{T}_+$. The Theorem says that $\mathbf{T}_{\Rsh c}$ is exact and such complexes "capture" any finite region of the Tate resolution.

Open questions 1

1. Are there Tate resolutions for sheaves on more general toric varieties?

   (It's easy to imagine a more general graded exterior algebra dual to the Cox ring; but the differentials seem to be of different "lengths". Perhaps a differential graded modules instead of a resolution?)

Open questions 1

1. Are there Tate resolutions for sheaves on more general toric varieties?

   (It's easy to imagine a more general graded exterior algebra dual to the Cox ring; but the differentials seem to be of different "lengths". Perhaps a differential graded modules instead of a resolution?)

2. If $\mathcal{F}$ is a sheaf on $\mathbb{P}^{n_1} \times \cdots \mathbb{P}^{n_t}$ that is a tensor product of sheaves pulled back from single factors (a "box product") then the Tate resolution is a true double complex.

   Does this property characterize direct sums of box products?

Open questions 2

3. Not only the corner comple’ but *many* “region” subcomplexes (some indices bounded below, some bounded above, some fixed) are exact!

   What do the region complexes mean?

Open questions 2

3. Not only the corner comple’ but *many* “region” subcomplexes (some indices bounded below, some bounded above, some fixed) are exact!

   What do the region complexes mean?

4. In an earlier paper we used Tate resolutions to prove that *every* complex of length m on $\mathbf{A}^n$ is the push-forward of a vector bundle on $\mathbf{A}^n \times \mathbb{P}^m$.

   Is every complex of coherent sheaves of length m on $\mathbb{P}^n$ the push-forward of a vector bundle on $\mathbb{P}^n \times \mathbb{P}^m$?