Puzzles Column

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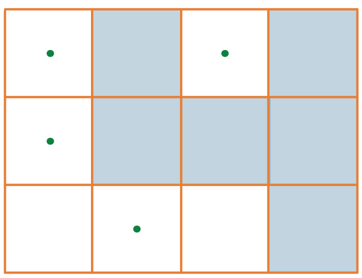
The 21st annual Bay Area Mathematical Olympiad took place on Feb 26, 2019. On March 10, MSRI hosted the awards ceremony, which also featured a special presentation by Mira Bernstein (Chair, Board of Directors, Canada/USA Mathcamp). Problems 3 and 4 below come from BAMO 2019. There are interesting things to say about the origins of the other problems: these will be included (along with solutions!) in the online version of this column at msri.org/emissary.

1. What is the smallest degree of a monic integer polynomial $f(x)$, such that 100 divides $f(n)$ for all integers n?

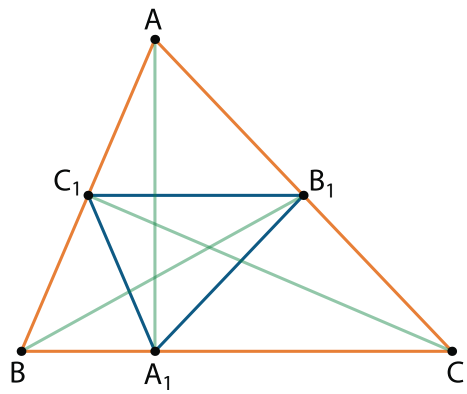
2. Flipping two coins produces three different results: "two heads," "two tails," and "one head and one tail." What is the expected number of these two-coin flips until all three results have been seen?

3. An 8×8 grid of squares starts with all squares being white. You choose any square and color it grey. Then you can repeat the following process for as long as you want: Choose any square that has exactly 1 or 3 grey neighbors, and color that square grey. (Two squares are neighbors if they share an edge.) Is it possible to end up with a grid in which every square is grey?



(To clarify which moves are allowed, in the smaller 3×4 grid above, six squares have been colored grey so far, and the only squares that can be colored now are marked with a dot.)

4. On a triangle ABC there are points A_1, B_1, C_1 on sides opposite the corresponding vertices. The line AA_1 is an altitude, BB_1 is a median, and CC_1 is an angle bisector. Prove that if the triangle $A_1B_1C_1$ is equilateral then so is ABC.



5. Prove that for every positive integer n, there are integers a and b such that

$$3^{3^n} - 1 = a^2 + b^2.$$

6. At time 0, point A is at $(0,0)$ in the plane, and point B is at $(0,1)$. Point A starts moving, at speed 1, to the right on the x-axis. Point B starts moving, at speed 1 and always towards A's position. What is the limiting distance between A and B?

2019 Mathical Book Prize Winners Announced

MSRI's Mathical Book Prize recognizes outstanding fiction and literary nonfiction for youth aged 2–18. The prize, now in its fifth year, is selected annually by a committee of pre-K-12 teachers, librarians, mathematicians, early childhood experts, and others. This year's winners are:

Pre-K: *Crash! Boom! A Math Tale* by Robie H. Harris (Candlewick Press)

Grades K–2: *Nothing Stopped Sophie: The Story of Unshakable Mathematician Sophie Germain* by Cheryl Bardoe (Little, Brown Books for Young Readers)

Grades 3–5: *The Miscalculations of Lightning Girl* by Stacy McAnulty (Random House Children's Books).

Grades 6–8: *To the Moon! The True Story of the American Heroes on the Apollo 8 Spaceship* by Jeffrey Kluger and Ruby Shamir (Philomel Books for Young Readers)

The winning titles were announced as part of the Critical Issues in Mathematics Education (CIME) conference at MSRI on March 7. As part of the 2019 National Math Festival, past and present Mathical award-winning authors will speak and host book signings for the public on May 4 in Washington, DC.

The Mathical Book Prize is awarded by MSRI in partnership with the National Council of Teachers of English and the National Council of Teachers of Mathematics, and in coordination with the Children's Book Council. The Mathical list is intended as a resource for educators, parents, librarians, children, and teens. Download the list at mathicalbooks.org.

Download a compete list of current and past winners and honorees at mathicalbooks.org.

Puzzle solutions follow on next page

Emissary Puzzles Column
Solutions, Spring 2019

June 30, 2019

Problem 1. What is the smallest degree of a monic polynomial $f(x)$, with integer coefficients, such that its integer values $f(n)$ are divisible by 100 for all integers n?

Solution: Let

$$f_d(x) := x(x-1)\cdots(x-d+1), \qquad \binom{x}{d} = \frac{x(x-1)\cdots(x-d+1)}{d!}$$

denote the "falling factorial polynomial" and "binomial coefficient polynomial" of degree d. The binomial coefficient polynomial takes integer values, and therefore all of the values of the falling factorial polynomial are divisble by $d!$. The smallest value of d such that $d!$ is divisible by 100 is $d = 10$, so the polynomial $f_{10}(x)$ is a monic polynomial with integer coefficients whose values are all divisble by 100.

This leaves the problem of showing that $d = 10$ is the smallest possible degree of such a polynomial. It is convenient to use finite differences. A finite difference table is the trapezoidal array that starts with a row of some number of initial values $f(0), f(1), \dots$ of a function (in our case, an integer polynomial of degree d, with at least $d+1$ values in the initial row), Subsequent rows are formed by subtracting two elements in the current bottom row (right minus left) and entering it in an offset new row below the current one. For instance, the first 5 rows of the finite diference table for the polynomial $f(x) = 2 - 3x + 5x^2 - x^3$, with 7 initial values, are:

$$\begin{array}{ccccccccccccc}
2 & & 3 & & 8 & & 11 & & 6 & & -13 & & -52 \\
& 1 & & 5 & & 3 & & -5 & & -19 & & -39 & \\
& & 4 & & -2 & & -8 & & -14 & & -20 & & \\
& & & -6 & & -6 & & -6 & & -6 & & & \\
& & & & 0 & & 0 & & 0 & & . & &
\end{array}$$

It is convenient to number the rows starting from 0; in the above table the third row is constant and the fourth row and all subsequent rows are all zero.

One proves by induction that the finite difference table of x^d has its d^{th} row constant, equal to $d!$, so that all later rows are all 0. By linearity it follows that the same statement

is true for any monic polynomial of degree d (the contribution of all x^k for $k < d$ dies out by the d^{th} row). In particular, if all of the first $d+1$ values of a monic integer-coefficient polynomial of degree d are divisble by 100 then $d!$ is divisble by 100, which shows that $d = 10$ is the smallest possible degree for a polynomial with the desired properties.
Comment: This specific question appears in a book "A Problem Seminar" by Donald Newman. Finite differences can be used show to prove the more general fact that the binomial coeffiucient polynomials are a basis of the ring of polynomials, with coefficients in the rational numbers, that take integer values at all integers.

Problem 2. Flipping two coins produces three different results: "two heads," "two tails," and "one head and one tail." What is the expected number of these two-coin flips until all three categories have been seen?

Solution: Let us denote different results for flipping two coins as HH for "two heads," TT for "two tails," and HT for "one head and one tail." The probability of HH or TT is $\frac{1}{4}$, while the probability of HT is $\frac{1}{2}$. Denote by p_S^n the probability that in the first n tosses we have only seen outcomes from a set S. If S contains only one element we get

$$p^n_{\{HH\}} = p^n_{\{TT\}} = \left(\frac{1}{4}\right)^n \qquad p^n_{\{HT\}} = \left(\frac{1}{2}\right)^n.$$

If S contains two elements we get

$$p^n_{\{HH,TT\}} = \left(\frac{1}{2}\right)^n \qquad p^n_{\{HH,HT\}} = p^n_{\{TT,HT\}} = \left(\frac{3}{4}\right)^n.$$

Denote by r_n the probability that we have not seen all three outcomes after the n-th toss, which we can calculate by using the principle of inclusion-exclusion:

$$r_n = p^n_{\{HH,TT\}} + p^n_{\{HH,HT\}} + p^n_{\{TT,HT\}} - p^n_{\{HH\}} - p^n_{\{TT\}} - p^n_{\{HT\}}.$$

Plugging in the values we get

$$r_n = \left(\frac{1}{2}\right)^n + 2\left(\frac{3}{4}\right)^n - \left(\frac{1}{2}\right)^n - 2\left(\frac{1}{4}\right)^n = 2\left(\frac{3}{4}\right)^n - 2\left(\frac{1}{4}\right)^n.$$

On the other hand, denote by q_n the probability that we first see all three outcomes after toss n. The problem asks to calculate

$$\sum_{n\geq 1} nq_n.$$

We can rearrange this into

$$\sum_{n\geq 1} q_n + \sum_{n\geq 2} q_n + \sum_{n\geq 3} q_n + \dots.$$

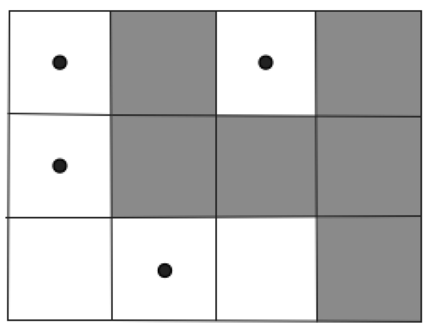
The nth sum above is the probability that we need at least n tosses, which is the same as the probability that we do not see all the outcomes in the first $n-1$ tosses. That means that the desired expectation is

$$\sum_{n\geq 0} r_n = 2\cdot 4 - 2\cdot\frac{4}{3} = 6\frac{1}{3}.$$

Comment: This problem is an instance of the Coupon Collector's Problem (CPP). The CCP can be formulated as following: Each box of a brand of cereals contains a coupon. There are n different types of coupons that are placed randomly in the boxes. The probability of buying the box with coupon i is p_i. What is the expected number of boxes a collector needs to buy to collect all n coupons?

Our problem corresponds to 3 coupons with probabilities $\frac{1}{4}$, $\frac{1}{4}$, and $\frac{1}{2}$.

Problem 3. An 8 by 8 grid of squares starts with all squares being white. You choose any square and color it grey. Then you can repeat the following process for as long as you want: Choose any square that has exactly 1 or 3 grey neighbors and color that square grey. (Two squares are neighbors if they share an edge.) Is it possible to arrive at the state in which all squares are grey?
(In the diagram, the marked squares have 1 or 3 neighbors.)

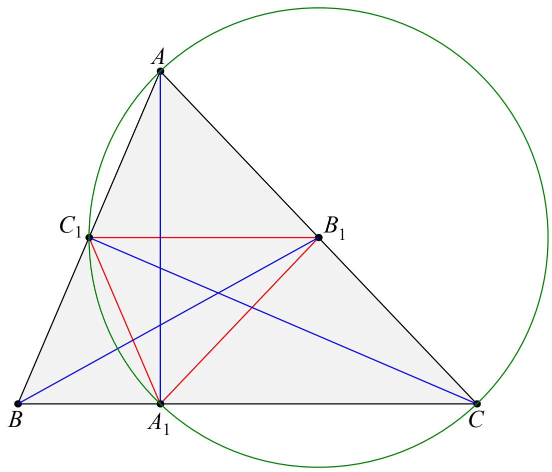


Solution: The fact that we can only add a square with an odd number of neighbors hints that there might be an invariant that helps to solve the problem.

Suppose the length of one edge in the grid is 1. Let $L(n)$ denote the length of the boundary of the gray region after n squares have been colored gray. Initially $L(1)=4$, since the perimeter of the first square colored is 4. If we add a square that has exactly one gray neighbor, the edge shared with that neighbor disappears from the boundary but is replaced by three new edges, so $L(n+1)=L(n)+2$. If we add a square that has exactly three neighbors, all three neighboring edges disappear from the boundary, but there will be one new edge, so $L(n+1)=L(n)-2$. In either case, knowing that $L(n)$ is even, we get $f(n)=\frac{L(n)}{2}+n$ modulo 2 is an invariant.

Since $f(1)$ is odd, then $f(n)$ must be odd for any n. On the other hand, all 64 squares are colored when $n=64$ and $L(64)=32$, which is the perimeter of the grid. Thus $f(64)$ is even. This is a contradiction, proving that it is not possible to color all the squares.

Problem 4. On a triangle ABC there are points A_1, B_1, C_1 on sides opposite the corresponding vertices. It is known that the line A_1A is an altitude, B_1B is a median, and C_1C is an angle bisector. Prove that if the triangle $A_1B_1C_1$ is equilateral then so is ABC.



Solution: Let s denote the length of the sides of the equilateral $\triangle A_1B_1C_1$. Since $\triangle AA_1C$ is right with midpoint B_1 on the hypotenuse AC, the point B_1 is the circumcenter of $\triangle AA_1C$ and

$$B_1A = B_1C = B_1A_1 = s.$$

As $B_1C_1 = s$, then C_1 belongs to the same circle centered at B_1. Therefore, $\triangle ACC_1$ is also right with hypotenuse AC. Thus CC_1 is an altitude in $\triangle AA_1C$, but since it is also an angle bisector, we conclude that $AC = BC$ and C_1 is the midpoint of AB. But since $\triangle AA_1B$ is also right, the median A_1C_1 is half of the hypotenuse AB. In other words,

$$AB = 2A_1C_1 = 2s = AC,$$

completing the proof.

Problem 5. Prove that for every nonnegative integer n, there are integers a and b such that

$$3^{3^n} - 1 = a^2 + b^2.$$

Solution: Applying the formula $x^3 - 1 = (x-1)(x^2+x+1)$ to $x = 3^t$ for $t = 3^{n-1}$gives

$$3^{3t} - 1 = \left(3^t - 1\right)\,\left(3^{2t} + 3^t + 1\right) = \left(3^t - 1\right)\,\left((3^t-1)^2 + 3^{t+1}\right).$$

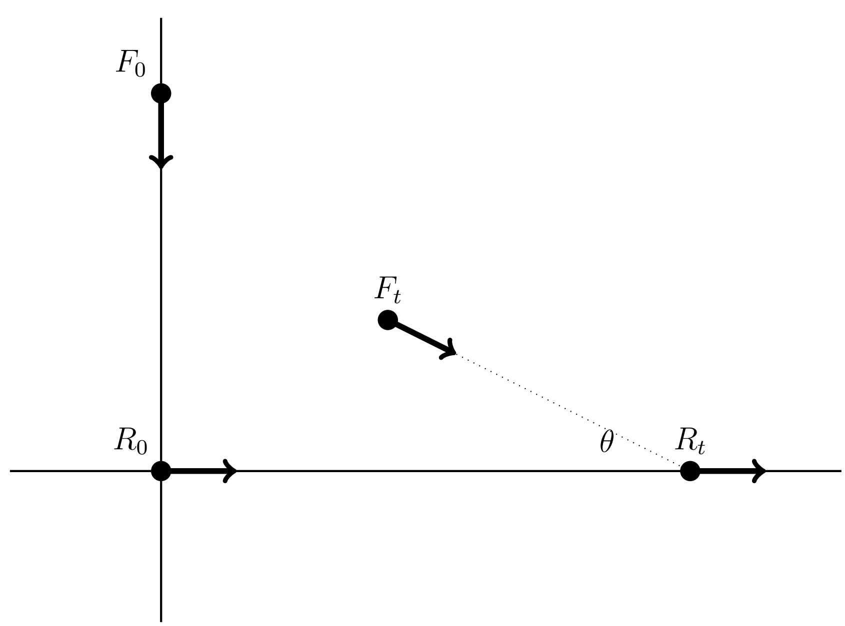
The second factor is overtly a sum of squares (note that $t + 1 = 3^n + 1$ is even). It is well known that the product of two sums-of-two-squares is again a sum of two squares, so the assertion follows by induction once the base case $n = 0$ has been verified:

$$3^1 - 1 = 2 = 1^2 + 1^2.$$

Comment: This curious result can be found in the paper *When is $a^n + 1$ the sum of two squares?* by Greg Dresden, Kylie Hess, Saimon Islam, Jeremy Rouse, Aaron Schmitt, Emily Stamm, Terrin Warren and Pan Yue, in the journal *Involve*, Vol. 12 (2019). We first saw it in Stan Wagon's PoW puzzle site.

Problem 6. At time 0 there is a rabbit at $(0, 0)$ in the plane, and a fox at $(0, 1)$. At time $t = 0$ the rabbit runs to the right with velocity 1 on the x-axis, and at the same time the fox runs directly towards the rabbit, and thereafter continues chasing the rabbit, also with velocity 1. It is easy to see that there is a limiting distance between the fox and the rabbit. What is that distance?

Comment: This is a famous example of a pursuit problem, and there is a vast literature on similar questions. Apparently Leonardo da Vinci thought about these, and the first mathematical results were obtained by Pierre Bouguer in 1732. Sergiu Hart posted this on his blog (http://www.ma.huji.ac.il/hart/puzzle/pursuit.pdf?), and Gil Kalai (https://gilkalai.wordpress.com/2018/06/29/test-your-intuition-35-what-is-the-limiting-distance) put it on his blog and, as is his habit, polled his readers for their quick guess as to the answer. The top two responses were 0 and 1, and the correct answer (1/2) came in third. We heard about this from Stan Wagon who put it on his email Problem-of-the-Week list, where Dan Velleman contributed an especially crisp solution.
Solution: In the diagram below, R_t and F_t denote the positions of the rabbit and fox at time t. The rabbit's position is $R_t = (t, 0)$ and the pursuing fox is at $F_t = (x, y)$. The velocity vectors are indicated in the diagram, and the angle between the x-axis and line segment from the fox to the rabbit is denoted θ. Throughout, $z = t - x$ will be the horizontal distance between the fox and rabbit, and $d = \sqrt{z^2 + y^2}$ will be the distance between them.



The intended insightful solution (which, arguably, does not require calculus) is as follows. In a small time interval $(t, t + dt)$ the distance between the fox and rabbit decreases by dt due to the fox's motion and increases by $\cos(\theta)\,dt$ due to the rabbit's motion. On the other hand, the horizontal distance z decreases by $\cos(\theta)\,dt$ due to the fox's motion and increases by dt due to the rabbit's motion. These are exactly opposite! It follows that $d + z$ is a constant.

Since y goes to 0 as t goes to infinity, it also follows that z approaches d, and since $d + z = 1$ (from the initial value) it follows that d and z both have the limit 1/2. (See Hart's blog, but also Mikhail Kagan's delightful article on the arXiv by with the title "Non-Cartesian Thinking.")

This is a very physicist-friendly solution, and putting it into equations gives a more mathematician-friendly version that also answers the lingering question of why y has to go to 0.

Differentiating the equations $z = t - x$ and $d^2 = z^2 + y^2$ gives

$$z' = 1 - x', \quad d' = \frac{zz'}{d} + \frac{yy'}{d}.$$

The fox is heading directly towards the rabbit so that

$$\frac{y'}{x'} = -\frac{y}{t - x} = -\frac{y}{z}.$$

Since the velocity $x'^2 + y'^2$ of the fox is equal to 1 it follows that

$$x' = \frac{z}{d}, \qquad y' = -\frac{y}{d}, \qquad z' = 1 - \frac{z}{d},$$

and therefore

$$d' = \frac{zz'}{d} + \frac{yy'}{d} = \frac{z}{d} - \frac{z^2}{d^2} - \frac{y^2}{d^2} = \frac{z}{d} - 1.$$

Comparing the formulas for d' and a' shows that $(d + z)' = d' + z' = 0$, as desired.

If y did not approach zero there would be a positive constant c such that $y > c > 0$. But then y's downwards speed $-y'$ would be bounded above 0 by

$$-y' = \frac{y}{d} \geq \frac{c}{d} \geq c$$

amd y would eventually go negative, which is impossible.