Random walks on weakly hyperbolic groups

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University of Toronto

Random and Arithmetic Structures in Topology
MSRI - Fall 2020

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Random walks, WPD actions, and the Cremona group

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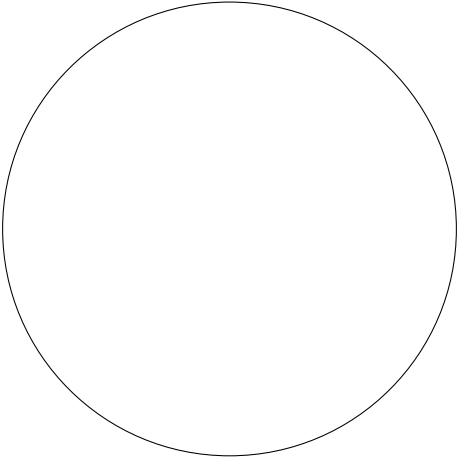

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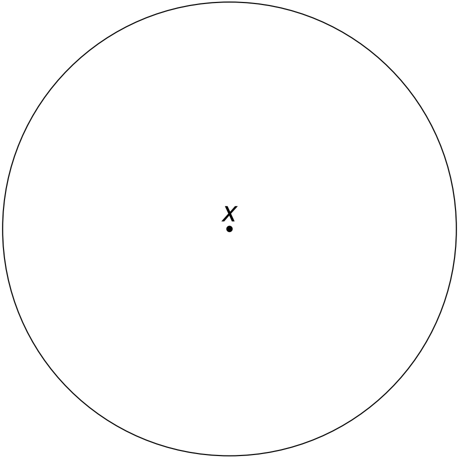

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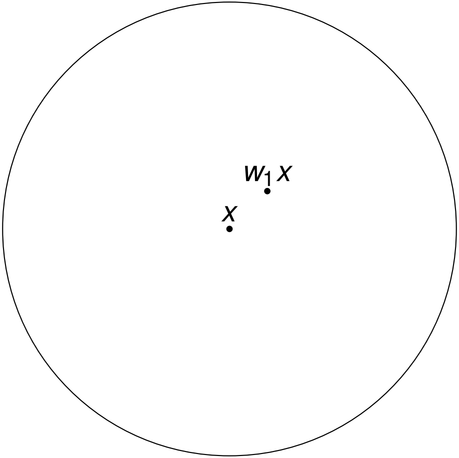

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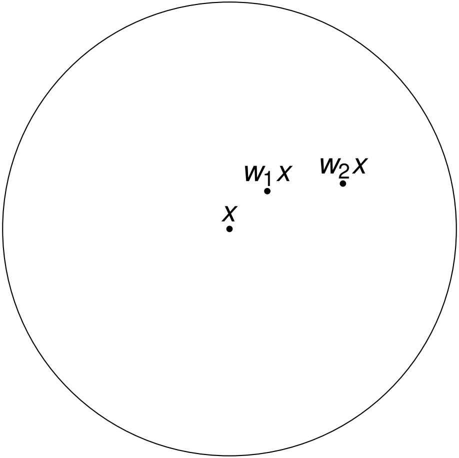

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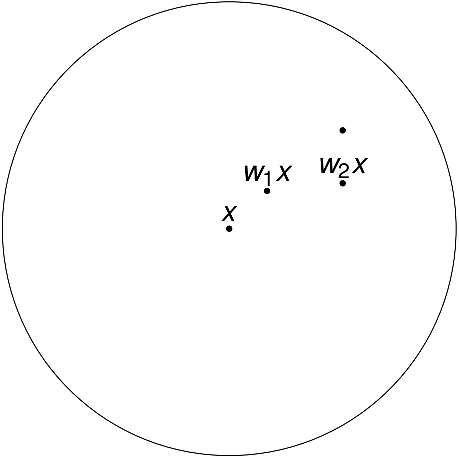

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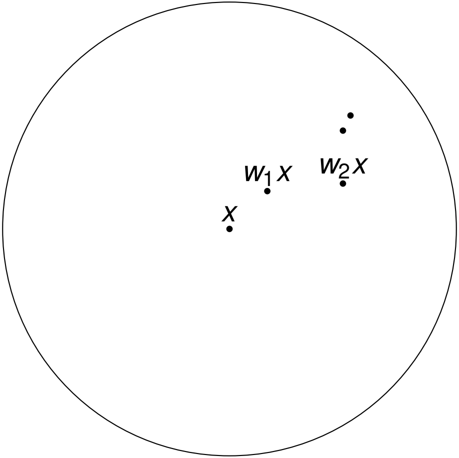

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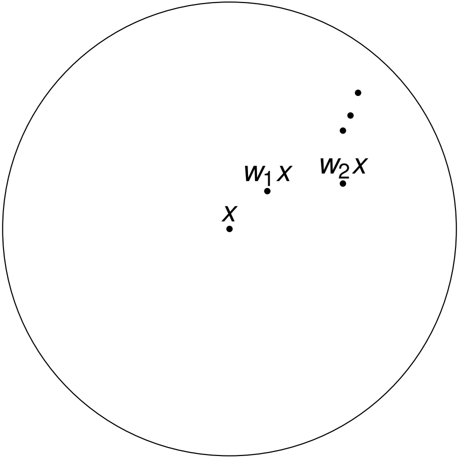

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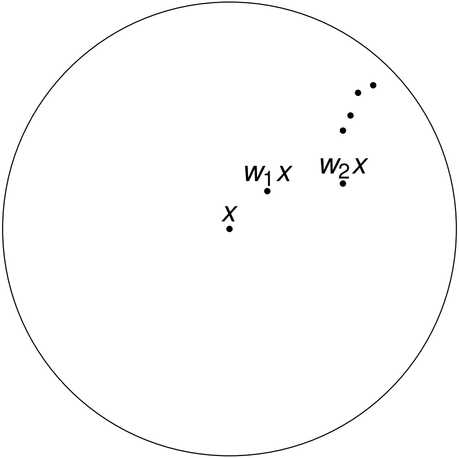

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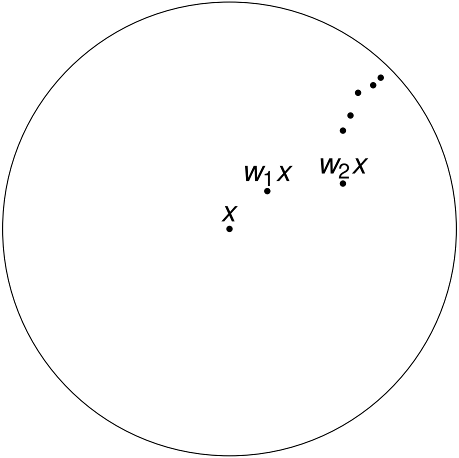

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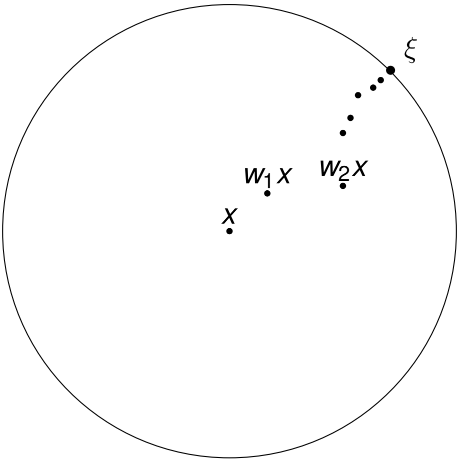

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Corollary.

$$\mathbb{P}(w_n \text{ is loxodromic }) \to 1$$

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2. *g has unbounded orbits and $\tau(g) = 0$. Then g is called parabolic.*
3. *$\tau(g) > 0$. Then g is called hyperbolic or loxodromic, and has precisely two fixed points on ∂X, one attracting and one repelling.*

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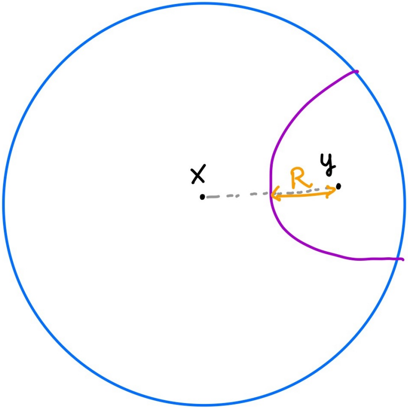

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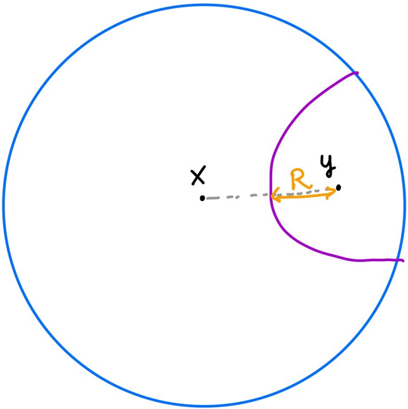

We call $r = d(x, y) - R$ the distance parameter.

Decay of shadows - I

Let us define

$$Sh(x, r) := \{S_x(gx, R) \ : \ g \in G, \ d(x, gx) - R \geq r\}$$

the set of shadows based at x, with centers on Gx and distance parameter $\geq r$.

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A shadow centered at gx of distance parameter r is contained in a ball of radius $\approx e^{-\epsilon r}$ in the metric d_ϵ on ∂X. As ν is non-atomic, the measure of a ball of radius $e^{-\epsilon r}$ tends to zero uniformly as $r \to 0$. $\square$

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for some $\varphi(r) \to 0$ as $r \to \infty$.

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$$x_n \in S_{x_{i+1}}(x_i, R) \text{ for all } n \leq i \tag{2}$$

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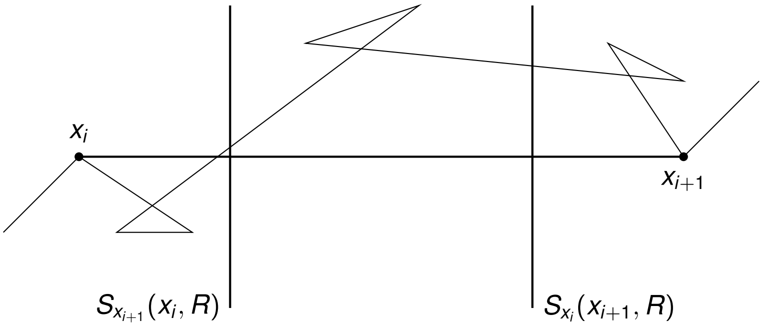

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Lemma

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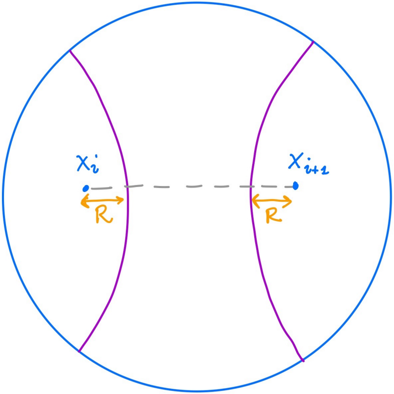

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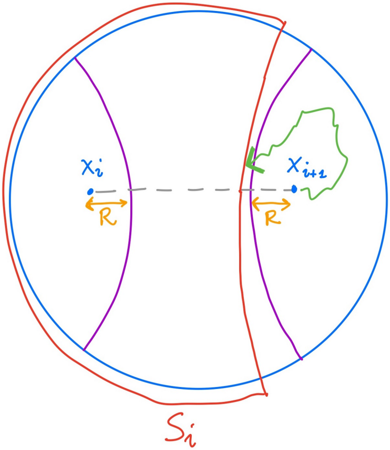

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which equals by the Markov property

$$1 - H_x^-(w_{ki}^{-1} S_i). \tag{4}$$

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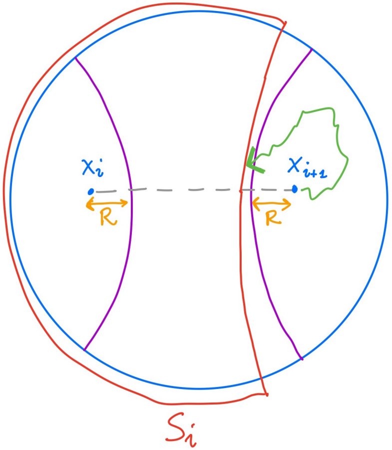
Given $\epsilon > 0$, there are R and k such that for any i each of (1), (2), (3) holds with probability at least $1 - \epsilon$.

The probability of (2) equals the prob. that $w_{kn}x$ never hits the complement of $S_{x_{i+1}}(x_i, R)$ for any $n \leq i$.



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The probability of (2) equals the prob. that $w_{kn}x$ never hits the complement of $S_{x_{i+1}}(x_i, R)$ for any $n \leq i$. As the complement of this shadow is contained in a shadow

$$S_i = S_{x_i}(x_{i+1}, R_i)$$

where $R_i = d(x_i, x_{i+1}) - R + O(\delta)$, the prob. that (2) holds is at least

$$1 - \mathbb{P}(\exists\, n \leq ki \;:\; w_n x \in S_i)$$

which equals by the Markov property

$$1 - H_x^-(w_{ki}^{-1} S_i). \tag{5}$$

The distance parameter of $w_{ki}^{-1} S_i$, is $R + O(\delta)$; hence, by decay of shadows, we may choose R sufficiently large such that (5) is at least $1 - \epsilon$.

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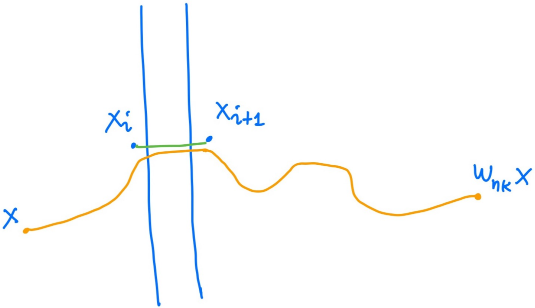

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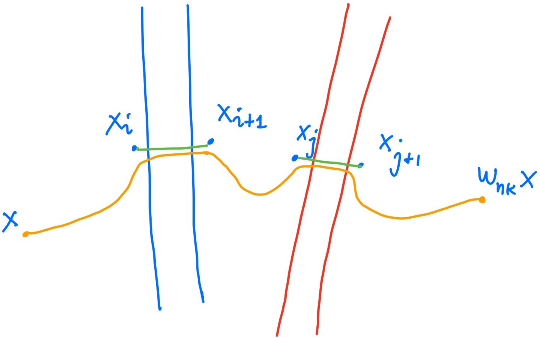

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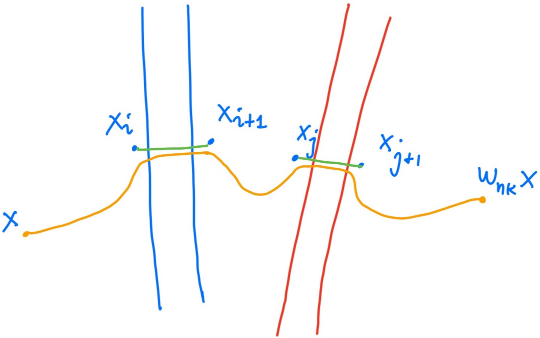

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- If $[x_j, x_{j+1}]$ is also persistent, then γ_i and γ_j are disjoint by (weak)-convexity of shadows.
- Therefore $d(x, w_{kn}x)$ is at least C times the number of persistent subsegments between x and $w_{kn}x$.

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then there is a U-invariant random variable W_∞ such that

$$\lim_{n\to\infty} \frac{1}{n} W_n = W_\infty$$

$\mathbb{P}$-almost surely, and in $L^1(\Omega, \mathbb{P})$.

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in L^1; finally, since $\mathbb{E}(Z_\infty) = \lim_n \mathbb{E}(\frac{1}{n} Z_n) \geq \eta > 0$, we have $A > 0$.

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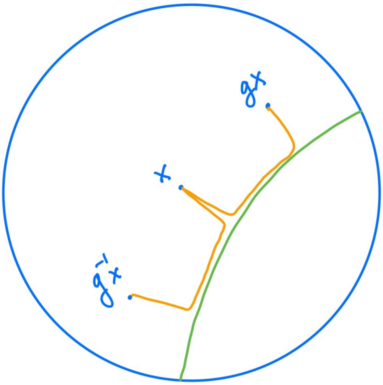

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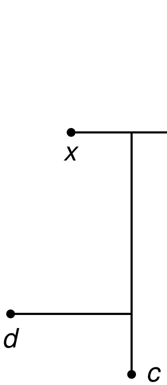

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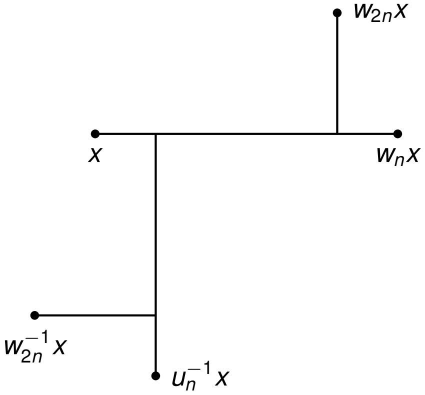

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Lemma (Lemma B)

For any $\ell < L/2$ we have

$$\mathbb{P}\left((w_n x \cdot w_{2n} x)_x \geq \ell n\right) \to 1, \text{ as } n \to \infty,$$

Lemma (Lemma B)

For any $\ell < L/2$ we have

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The same argument applied to $w_{2n}^{-1} x$ shows

$$\mathbb{P}\left((u_n^{-1} x \cdot w_{2n}^{-1} x)_x \geq \ell n\right) \to 1$$

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$$(w_{2n}^{-1}x \cdot w_{2n}x)_x = o(n)$$
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$$\tau(w_{2n}) = d(x, w_{2n}x) - 2(w_{2n}^{-1}x \cdot w_{2n}x)_x + O(\delta) \geq (L-\epsilon)(2n)$$
which completes the proof.