Overview of multivalued solutions in differential geometry

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"Multivalued" objects (functions, forms, spinors, ...)
Example $\sqrt{z}$ on **C**.

Defined as

- function on cut plane;
- section of a flat bundle over $\mathbf{C} \setminus \{0\}$ with holonomy -1;
- function on branched cover.

Section 1. Some classical topics (real dimension 2, Riemann surfaces)

I. **Submanifolds** (complex curves).

$C \subset S$ a smooth curve in a complex surface.

$C_i \subset S$ a sequence of curves converging to C. Modelled on sections s_i of the normal bundle $N \to C$.

If instead C_i converge to $2C$ (multiplicity 2), the models are multivalued sections of N, i.e. $s_i + \sqrt{\sigma_i}$ for sections σ_i of $N^{\otimes 2}$.

II. **Periods** Let $\mathcal{X} \to D$ be a family of n-dimensional varieties over the disc such that the central fibre X_0 has an ordinary double point singularity. The cohomology $H^n(X_t)$ of the fibres defines a flat vector bundle E over $D \setminus \{0\}$. Holonomy given by the Picard-Lefschetz formula

$$\alpha \mapsto \alpha + (\delta.\alpha)\delta,$$

where $\delta \in H_n$ is the "vanishing cycle". For n even, the holonomy takes δ to $-\delta$.

Suppose that Ω is a holomorphic $(n+1)$-form on the total space $\mathcal{X}$. Then the contraction with $\frac{\partial}{\partial t}$ defines a family ω_t of holomorphic n-forms on the X_t— a section of E. For example, the periods

$$\int_\delta \omega_t,$$

yield a multivalued function f on D.

More invariantly, this is a multivalued holomorphic 1-form $f(t)dt$.

III. **Gauge theory**

Hitchin's equations for an $SU(2)$-bundle $P \to S$, where S is a Riemann surface.

$SU(2)$-connection A, section $\psi \in \Omega^1(\mathrm{ad}\, P)$,

- $d_A^* \psi = 0$;
- $d_A \psi = 0$;
- $F_A = [\psi, \psi]$.

Kobayashi-Hitchin correspondences with

- Stable flat $SL(2, \mathbf{C})$ connections;
- Stable pairs (E, ϕ): E a holomorphic vector bundle and $\phi \in \Omega^{1,0}(\mathrm{End}\, E)$ holomorphic.

For a pair (E, ϕ) we have $\sigma = \det \phi$, a holomorphic quadratic differential on S.
Locally, away from the zero set, $E = L \oplus L^{-1}$ and

$$\begin{pmatrix} \sqrt{\sigma} & 0 \\ 0 & -\sqrt{\sigma} \end{pmatrix}.$$

Differential geometry: consider scaled version $\epsilon F_A = [\psi, \psi]$ with parameter $\epsilon \to 0$ (Mazzeo,Swobada, Weiss, Witt ...).

In the limit $[\psi, \psi] = 0$. Locally, away from the zero set, $\psi = \lambda \otimes v$ for a unit section v of $\mathrm{ad}\, P$ and harmonic 1-form λ.

But v, λ are only defined up to sign, so λ is a multivalued harmonic 1-form.

(In fact $\lambda = \mathrm{Re}\sqrt{\sigma}$.)

Section 2: Higher dimensions

More recent developments, in particular work of Taubes from 2012, consider multivalued objects on an n-dimensional manifold M with branching on a codimension 2 submanifold $\Sigma \subset M$. These include higher dimensional analogues of each of the topics I,II, III.
Common features:

- *geometry* is locally modelled on the 2-dimensional case, transverse to Σ;
- *analysis* is more complicated because there is an infinite dimensional space of deformations of Σ, and in some situations Σ might have singularities.

Gauge theory: Taubes analysed limits of flat $SL(2, \mathbf{C})$ connections over a 3-manifold M (and more generally data which satisfy the equation up to term bounded in L^2). The limit defines a multivalued harmonic 1-form on $M \setminus \Sigma$.

Similar results for other equations in 3 and 4 dimensions. (Taubes, Haydys, Walpsuski, Zhang ...). For example, Taubes (arxiv 1702.04610) studied limits of solutions to the Vafa-Witten equations on a 4-manifold X, which lead to multivalued self-dual harmonic 2-forms on $X \setminus \Sigma$.
See also the talk of Greg Parker in this workshop. There are also connections with gauge theory in higher dimensions (Haydys, Walpuski).

Calibrated submaifolds

Siqi He constructed sequences of Special Lagrangian submanifolds of a Calabi-Yau manifold converging to a multiplicity 2 limit using multivalued harmonic 1-forms. See the talk of Siqi He in this workshop.

Related work of Doan and Walpuski in the context of enumerative theories, for 3-dimensional associative submanifolds in 7-dimensional G_2 manifolds.

Perhaps also for 4-dimensional coassociative submanifolds and Cayley submanifolds of 8-dimensional $\mathrm{Spin}(7)$-manifolds.

There are intriguing connections between these developments. For example the problem of defining Vafa-Witten invariants on general 4-manifolds might well be relevant to the problem of counting coassociative subamifolds in G_2-manifolds.

Section 3: Multivalued harmonic 1-forms: some analysis foundations

Consider a compact Riemannian n-manifold (M, g), a co-oriented codimension 2 submanifold $\Sigma \subset M$ and a flat **R**-line bundle $E \to M \setminus \Sigma$ with holonomy -1 around Σ.

We want to study bounded harmonic 1-forms λ on $M \setminus \Sigma$ with values in E. Locally these are derivatives of E-valued harmonic functions. More generally they can be identified with harmonic sections u of an affine extension of E.

Such a 1-form λ has a single-valued lift $\tilde{\lambda}$ to a double branched cover $\tilde{M} \to M$. This defines a de Rham cohomology class in $\mathbf{H}_E = H^1_-(\tilde{M})$, the -1 eigenspace of the involution on $H^1(\tilde{M})$.
The lift of the metric g to $\tilde{M}$ is not a smooth Riemannian metric but a version of the Hodge Theorem applies, so any class c in $\mathbf{H}_E$ has a harmonic representative $\tilde{\lambda}$ which corresponds to a λ on M.
The problem is that for general data (g, Σ, c) the 1-form λ will not be bounded.

Local model.

Let z be a coordinate transverse to Σ and y a coordinate along Σ. Then $\lambda = du$ where, as $z \to 0$,

$$u \sim \mathrm{Re}(A(y)z^{1/2} + B(y)z^{3/2}).$$

Globally, A is a section of $\nu^{-1/2}$ and B is a section of $\nu^{-3/2}$, where $\nu \to \Sigma$ is the normal bundle.

For du to be bounded we need $A = 0$.

To achieve this we need to choose Σ depending on the other data (g, c).

Deformation Theorem *Suppose that we have a solution* λ_0 *for data* (g_0, c_0) *with branch set* Σ_0*. Suppose that the subleading term* B *is nowhere* 0 *on* Σ_0*. Then for any* (g, c) *close to* (g_0, c_0) *there is a submanifold* Σ *close to* Σ_0 *such that a bounded solution* λ *exists.*

A similar result was obtained by Takahashi in the case of harmonic spinors.

Let $\mathcal{S}$ be the space of all submanifolds Σ and $\mathcal{V} \to \mathcal{S}$ the infinite-dimensional vector bundle with fibres $\Gamma(\nu^{-1/2})$. So we can think of A as defining a section of $\mathcal{V} \to \mathcal{S}$. The derivative, at Σ_0, is defined by the pairing with B under the map

$$\nu \otimes \nu^{-3/2} \to \nu^{-1/2},$$

so the hypothesis on B implies that this is surjective.

Thus the statement is of implicit function theorem type.

It seems that the implicit function theorem in Banach spaces (Sobolev, Hölder, . . .) does not suffice due to loss of derivatives of the submanifolds.

But a version of the Nash-Moser theory covers the situation.

Remark in the "classical case", when $n = 2$, this theorem becomes a well-known statement about quadratic differentials.

Section 4: Adiabatic G_2-structures

A nonlinear variant of the harmonic 1-form equation

Let E be a flat bundle over an n-manifold M_0 with fibre $\mathbf{R}^{n,m}$ (i.e the structure group is $O(n,m)$) and let L be a 1-form on M_0 with values in E. At each point $x \in M_0$ we have a map

$$L_x : TM_x \to E_x .$$

Suppose that the image of this map is a maximal positive subspace, with respect to the quadratic form on the fibre. Then we get a Riemannian metric γ on M_0 (so that the L_x are isometries).

We have a system of PDE for L:

1. $dL = 0$;
2. $d^*_\gamma L = 0$.

where d^*_γ is the operator defined by the metric γ (which also depends on L). So this is a nonlinear PDE for L.

These equations for L are locally equivalent to the "maximal submanifold" equations for positive 3-dimensional submanifolds in $\mathbf{R}^{3,19}$.

That is, equation (1) means that we can write $L = dU$, locally, for a map U into $\mathbf{R}^{3,19}$ and the submanifold is the image of this map.

Motivation

A torsion-free G_2-structure on a 7-manifold Y can be defined by a closed 3-form ϕ satisfying certain conditions (a "positivity" condition and a PDE).

A 4-dimension submanifold $N \subset Y$ is coassociative if $\phi|_N = 0$.

Consider a 7-manifold Y and a smooth map $\pi : Y \to M$ to a 3-manifold M which is a fibration outside a codimension 2 submanifold $\Sigma \subset M$ (i.e. a link).

We suppose that for $x \in M \setminus \Sigma$ the fibres $\pi^{-}(x)$ are diffeomorphic to the K3 manifold N, and that over Σ the fibres have ordinary double point singularities.

A particular case is for $Y = Z \times S^1$ and $M = S^2 \times S^1$ with the product of a Lefschetz fibration $Z \to S^2$ and a trivial S^1-factor.

We have a flat cohomology bundle $E \to M \setminus \Sigma$ with fibre $H^2(N) = \mathbf{R}^{3,19}$ (the cup product form on H^2).

General fact:
Any closed 3-form on Y which vanishes on the fibres of π defines a closed 1-form L on $M \setminus \Sigma$ with values in the flat bundle E.
(One way of seeing this: locally, over an open set $\pi^{-1}(U)$ for $U \subset M \setminus \Sigma$ we can write $\phi = d\rho$. Then on each fibre ρ is a closed 2-form which defines a section of E. The derivative of this section is the desired E-valued 1-form.)

One can show/argue that the *adiabatic limit* of the G_2-equations for a manifold Y with a coassociative fibration π, as the volume of the fibre tends to zero, is the above system of equations for L.

The main ingredient is the Torelli theorem for hyperkähler metrics on the K3 manifold N.

We can abstract the whole set-up and consider a flat vector bundle $E \to M \setminus \Sigma$ where the fibres have a form of signature $(3, q)$ and with Picard-Lefschetz monodromy around Σ. A closed E-valued 1-form L defines a class c in a certain cohomology group $\mathbf{H}_E$. The general question is when does there exist an L in a given class c satisfying the equation $d_\gamma^* L = 0$.

In the case when $Y = Z \times S^1$ this becomes a question of Torelli type for Lefschetz fibred Calabi-Yau structures on Z.

There is a deformation theorem, with respect to variations in c, similar to that in Section 3 above. The nonlinearity of the PDE introduces extra difficulties.

More challenging, for the future, is to understand what can happen for "large variations" in c.

Examples of phenomena that might occur:

- The diameter of M in the metric γ goes to infinity. (Twisted connected sums ?)
- Two components of the link Σ come together. (Bryant-Salamon cones over $S^3 \times S^3$)?)
- A component shrinks to a point. (Bryant-Salamon cones over $\mathbf{C}¶^3$?)

There are also various interesting connections with other topics we have discussed in this talk (gauge theory, calibrated geometry, enumerative theories . . .).