A. Beilinson : Langlands Correspondence in the deRham setting 2

$\mathcal{A}_X \boxtimes \mathcal{A}_X \longrightarrow \Delta_* \mathcal{A}_X$

$a \boxtimes b \longmapsto a \circ b \in \mathcal{A}_X((t_1 - t_2))$

fixed curve X. Y scheme $(JY)_x = \mathrm{Hom}(\mathrm{Spec}\, \mathcal{O}_x, Y)$

$\Rightarrow$ factorization space $(JY)_{(x_1,\dots x_n)} = \mathrm{Hom}(\mathrm{Spec}\, \mathcal{O}_{(x_1,\dots x_n)}, Y)$

schemes over powers of X with compatibilities at and away from diagonals - can replace by ind-schemes etc

Instead of jets can consider meromorphic jets : localize

Y affine $\Rightarrow$ ind scheme $\mathcal{J}^{mero} Y$: $\mathcal{O}_{(x_1,\dots x_n)} \rightsquigarrow F_{(x_1,\dots x_n)}$

invert equations of points.

For chiral algebras / factorization / Vertex algebra can define modules:

A algebra, to give module structure on M is same as giving algebra structure on $A \oplus M$ restricting to 0 on M, usual on A, action on (a, m)

$\longrightarrow$ obvious notion of modules

$\mathcal{A}$ chiral algebra

$x \in X \Rightarrow$ category $\mathcal{M}(\mathcal{A})_x$ of modules supported at x

M module $\Leftrightarrow$ $\mathcal{A}_{\mathcal{O}_x} \boxtimes M \longrightarrow F_x \hat{\otimes} M$

F_x = local field. vector space $\rightsquigarrow$ vertex operator

$\Leftrightarrow$ can replace $\mathcal{A}_{\mathcal{O}_x}$ by $\mathcal{A}_{F_x}$: only depends on $\mathcal{A}$ on punctured disc (satisfying some properties..)

Can twist chiral algebras wrt torsors :

Chiral alg form tensor category, clear from factorization (just tensor your sheaves)

G_X group scheme / X with connection - in particular commutative chiral algebra (jets). Can speak of G_X action on chiral algebra $\mathcal{A}$

$\mathcal{F}_X$ G_X torsor (plain torsor + connection)

$\Longrightarrow$ can twist $\mathcal{A}$ by $\mathcal{F} \Rightarrow \mathcal{A}^{\mathcal{F}}$

Particular case : constant group scheme. torsor = plain G-torsor

Another case : G any group scheme / X $\Longrightarrow$ JG_X jets has connection, JG torsors (w. conn) $\Longleftrightarrow$ G-torsors !

[Note $G_x \subset JG_x$.]
[Correction from last time: $\Gamma_{\mathcal{L}}$: $\mathcal{D}$-mod on affine flag $\to$ reps
$\mathcal{L}$ ample lying in orbit ... no choices involved!]

The affine Grassmannian $\quad G$ reductive group, X curve

$$\mathcal{G}r_x = G(F_x)/G(\mathcal{O}_x)$$

F_x: Laurent at x, $\mathcal{O}_x$: Taylor at x

gives ind-scheme $/X$ with these as fibers.

$\mathcal{G}r_X$ carries canonical factorization structure & connection & unit section

integral quadratic forms $\Rightarrow$ line bundle on $\mathcal{G}r$, also carries natural factorization structure.

Level c $\Rightarrow$ line bundle $\mathcal{L}_c$ on $\mathcal{G}r_{X^n}$ every n compatible with factorization!

eg on X^2: on diagonal have $\mathcal{G}r_X$ with $\mathcal{L}_c$
off diagonal have $\mathcal{G}r_X \times \mathcal{G}r_X$ with $\mathcal{L}_c \boxtimes \mathcal{L}_c$

Claim: Let P be a commutative algebra with action of $G^\vee$ $\Rightarrow$ canonical assignment of vertex algebra on X, $\mathcal{A} = \mathcal{A}(P)$

Fibers: P as $G^\vee$-module $\Rightarrow$ perverse sheaf on $\mathcal{G}r$ given by Satake equivalence
(P not nec fin-dim but algebraic — sum of f.d. reps, so perverse sheaf will be sum of basic IC sheaves)

$\mapsto$ $\mathcal{D}$-module $\mathcal{P}_X$ on $\mathcal{G}r_X \longmapsto \Gamma(\mathcal{G}r_X, \mathcal{P}_X \otimes \mathcal{L}_c) = \mathcal{A}_X$

global sections of twisted $\mathcal{D}$-module. $\quad c$ a subcritical weight $\Rightarrow$ no higher cohomology

$\mathcal{A}_X$ is in fact a factorization algebra, thanks to Satake:
eg define $\mathcal{A}_{X^2}$

$$\begin{tikzcd}
\mathcal{G}r_X \times \mathcal{G}r_X|_U \arrow[r, hook] \arrow[d] & \mathcal{G}r_{X^2} \arrow[d] & \mathcal{G}r_X \arrow[l, hook', "\Delta"'] \arrow[d] \\
U \arrow[r, hook] & X^2 & X \arrow[l, hook']
\end{tikzcd}$$

Use algebra structure on $\mathcal{P}$: $Perv_{G(\mathcal{O})}(Gr)$ is a tensor category, $Rep_G \longrightarrow$ tensor functor, so $\mathcal{P}$ gives algebra wrt this tensor product, $\mathcal{P}_X$

Take $j_* j^* \mathcal{P}_X \otimes \mathcal{P}_X \longrightarrow \Delta_* \mathcal{P}$ way to rewrite product of $\mathcal{P}$

Take the <u>kernel</u> of this map & push forward

$\pi_* Ker = \mathcal{A}_{X^2}$.

$\mathcal{P}$ trivial (unit) algebra $\Longrightarrow$ vaccuum representation of Kac Moody algebra. $\mathcal{P} = \delta$-functions on distinguished point of Gr

We'll consider $\mathcal{P}$ = regular rep, as left $G^\vee$-module $\Rightarrow$ right action gives algebra with $G^\vee$-action in $Rep\, G^\vee$ $\Rightarrow$ get chiral algebra with $G^\vee$ action.

<u>Def</u> $\mathcal{A}$ corresponding to $\mathcal{P}$ = regular rep is called the <u>chiral Hecke algebra</u>, carries $G^\vee$ action $\Rightarrow$ can twist by any $G^\vee$-torsor $\mathcal{F}$ with connection, & consider category of chiral modules supported at a point:

$x \in X$ $\mathcal{F}$ $G^\vee$-loc system on $Spec\, F_x$ $\rightsquigarrow \mathcal{A}^{\mathcal{F}} \rightsquigarrow$ $\mathcal{M}(\mathcal{A}^{\mathcal{F}})_x$ - only need $\mathcal{F}$, $\mathcal{A}^{\mathcal{F}}$ on <u>punctured</u> disc to define this category of modules supp at point.

As $\mathcal{F}$ varies get family of chiral algebras over $\mathcal{LS} = \mathcal{LS}(G^\vee, Spec F)$ moduli of local system on $Spec\, F_x$.

trivial $\subset$ regular rep $\Longrightarrow$ Kac-Moody sits $G^\vee$-invariants in $\mathcal{A}^{\mathcal{F}}$, so $\mathcal{A}^{\mathcal{F}}$-module gives K-M module - consider as family of KM modules over $\mathcal{LS}$

$$\Gamma : \mathcal{M}(\mathcal{A}^{\mathcal{LS}})_x \longrightarrow \mathfrak{g}(F_x)^c \text{-modules}$$

<u>Conjecture</u> : This is an equivalence of categories

$T(F_x) = \Gamma \otimes F_x^*$

<u>$G = T$ torus</u> level $\iff$ $\mathbb{Z}$-valued bilinear form on lattice Γ corresp. to T. Here just need <u>nondegen</u> form (no negativity) $\Rightarrow$ lattice Heisenberg vertex algebra:

$T(F_x)$ group indscheme, $\iff$ (almost) canonical extension of $T(F_x)$ by $\mathbb{G}_m$, the Heisenberg group. $T(F_x)^c$.
Commutator pairing $T(F_x) \times T(F_x) \longrightarrow \mathbb{G}_m$
$$[\gamma_1 \otimes f_1, \gamma_2 \otimes f_2] = \{f_1, f_2\}^{-c(\gamma_1, \gamma_2)}$$

$\{\ \}$ Tame symbol with parameters (Contou-Carrere

lattice not even $\Rightarrow$ <u>super</u> extension by $\mathbb{G}_m$.
extra structures: splitting $T(\mathcal{O}_x) \subset T(F_x)^c$
& symmetric structure: inverse involution on $T(F_x)$ lifts to Heisenberg.

$\mathrm{Ind}_{T(\mathcal{O}_x)}^{T(F_x)^c}(1)$ = fiber of lattice Heisenberg structure $\mathcal{A}$

Connected component of $0 \in \Gamma$ $\Rightarrow$ <u>Lie algebra</u> induced rep = vacuum rep of Heisenberg Lie algebra $\mathcal{A}^0 < \mathcal{A}$.

Modules for a vacuum rep of $\mathfrak{g}$ $\iff$ modules of $\mathfrak{g}$

Reps of $\mathcal{A}$ is a small, semisimple category: unlike those of $\mathcal{A}^0$ much nastier.
Our form $c: T \longrightarrow T^\vee$ has finite kernel Z
$\mathcal{A}$-mod $\overset{\sim}{\longleftrightarrow}$ Z-mod reps of finite group scheme!

<u>Twists & rigidity</u> $\mathcal{A}$ is Γ-graded $\iff$ has $T^\vee$ action
$T(\mathcal{O}_x)$ action (<u>induced</u> rep of Heisenberg)
- jet group scheme of T
- action of 2 group schemes on our chiral algebra, one of fin type (with connection), one of jet type

(Same true for $G(\mathcal{O})$, $G^\vee$ on chiral Hecke)

but here they're compatible under $Z \hookrightarrow T(\mathcal{O}_x) \overset{c}{\longrightarrow} T^\vee(\mathcal{O}_x)$

$$\begin{array}{ccccc} & & \cup & & \cup \\ Z & \hookrightarrow & T & \overset{c}{\longrightarrow} & T^\vee \end{array}$$

Z = also kernel $T(\mathcal{O}_x) \to T^\vee(\mathcal{O}_x)$

So action is trivial on Z, & action of constant group $T^\vee$ extends to action of $T^\vee(\mathcal{O}_x)$
So twist by $T^\vee$ local system $\longleftrightarrow$ extend to $T^\vee(\mathcal{O}_x)$ - torsor (inducing) & twist by this larger torsor.

BUT $T^\vee(\mathcal{O}_x)$ - torsor + conn $\Longleftrightarrow$ $T^\vee$ bundle :
result independent of connection on $T^\vee$ - bundle!

$$\begin{array}{ccc} \mathcal{A}^F & \xrightarrow{\sim} & \mathcal{A} \\ \cup & & \cup \\ \mathcal{A}^0 & \xrightarrow{\sim} & \mathcal{A}^0 \end{array}$$

depending on trivialization of $T^\vee$-bundle

not identity : depends on potential of connection
So $\mathcal{A}$ doesn't change but rep of $\mathcal{A}^0$ will change, by action of Heisenberg group changing connection

$\Rightarrow$ implies conjecture for $G = T$: look at $\mathcal{L}S_{T^\vee}$
- for fixed $T^\vee$ bundle eg triv connections $\longleftrightarrow$ one-forms, take mod gauge transformations of $\mathcal{O}^*$
- so only polar part of forms survive, write as sum of residue + purely irregular part.

Group acts by translation by integers — acts infl simply transitive

$\Rightarrow$ looks like $\mathbb{A}^1/\mathbb{Z} \times \{\omega(F)/\omega^{\leq 1}\}/_{\text{infl translation}} \times B\mathbb{G}_m$ (poles order ≤ 1)

- Consider $\mathcal{O}$-modules on this : on $\mathbb{A}^1$ $\mathbb{Z}$-equivariant, on pole part get $\mathcal{D}$-mod, + extra grading from $B\mathbb{G}_m$.
Fiber at point is abelian semisimple category with fin many simple objects

On other hand Heisenberg reps : $F^c = \mathbb{C} \times \mathfrak{m} \times$ polar part (pair nondegenerately)
modules over Heisenberg $\longleftrightarrow$ $\mathcal{O}$-mod on $\mathbb{A}^1 \times \{\omega(x)/\omega^{\leq 1}\}/_{\text{infl translation}}$

... $\mathbb{Z}$ & $B\mathbb{G}_m$ cancel each other

"Global sections" on our nonalgebraic stack : define by hands as coinvariants ...