A. Braverman Formal arcs to algebraic semigroups & automorphic L-functions

Understand geometrically local L-functions

Motivation 1. Can speak of in following setting: K local field, $q = \#$ res field
G split reductive group $/K$, $\rho: G^\vee \to GL(n)$
$\rightsquigarrow L(\rho, \pi, s) \in \mathbb{C}(q^s)$ π : irrep of G.

eg. • $G = GL(n)$ ρ = standard well understood
• ρ arbitrary, π unramified "

$\pi \in Irr\ G(K)$ $G = GL(n)$, take $\int_G c_\pi(g) f(g) |\det g|^s dg$
c_π : matrix coefficient, $\cap$
$f \in C_c^\infty(Mat(n, K))$ $\mathbb{C}(q^s)$
$=: S(Mat(n,K))$
all such integrals form a fractional ideal J_π of $\mathbb{C}[q^s, q^{-s}]$,
$L(\pi, s)$ = common denominator of these integrals (all c_π, f)

Need: generalize Schwartz-Bruhat space $S(Mat(n,K))$
- as functions coming from perverse sheaves via fonctions ↔ faisceaux

In general for $\rho: G^\vee \to GL(n)$ make positivity assumption:
$\exists\ \sigma: \mathbb{G}_m \xrightarrow{\sigma} G^\vee \hookrightarrow GL(n)$ [σ lies in center of $G^\vee$]
s.t. $\rho\circ\sigma$ is sum of characters $t \mapsto t^i$ for $\underline{i > 0}$.

∴ in particular $G^\vee$ $\underline{\text{not}}$ semisimple, more like GL_n.

π unramified: (for simplicity assume only wt 1 of $\mathbb{G}_m$ occurs in ρ)
$\pi \rightsquigarrow a \in G^\vee$ s.s. conj class

$$L(\rho, \pi, s) = \frac{1}{\det(1 - q^{-s}\rho(a))} \quad \text{local } L\text{-factor.}$$

[eg standard rep of $GL_n \supset \mathbb{G}_m$ central $\Rightarrow \forall i = 1$]
s in L-fn "always comes from such a center ...

GL_n case: e = char. function of $Mat(n, \mathcal{O}) \subset Mat(n,K)$
take as our f in $\int$

Want $S_\rho(G)$ space of fns on G (not with compact support on G: e.s. compact support on $Mat \supset GL$)

Have $c_\rho \in S_\rho(G)$ $G(O)\times G(O)$-invariant
- by Satake corresponds to a rep of $G^\vee$ -
which up to q's corresponds to rep $Sym^*(\rho) = \bigoplus Sym^i \rho$

Positivity condition will guarantee that as function this infinite sum is well defined

$\boxed{\text{Goal}}$: give geometric interpretation of c_ρ.

c_ρ in a spherical rep multiplies spherical vector exactly by L-function: π spherical rep $\ni v_\pi$ spherical vector $\Rightarrow$ $c_\rho \cdot v_\pi = L(\rho, \pi, 0)\cdot v_\pi$.

eg standard rep $\Leftrightarrow$ char function of $Mat_n(O)$

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2. Grinberg-Kazhdan-Drinfeld on formal arcs:
X alg variety $/k$, $\mathcal{L}(X)$ = scheme of formal arcs in k $\longleftrightarrow$ maps $D = \mathrm{Spec}\, R[[t]] \longrightarrow X$ (R-points)

$\gamma \in \mathcal{L}(X)(k)$, look at formal neighborhood of γ $\widehat{\mathcal{L}(X)}_\gamma$

Theorem (G-K char $=0$, Drinfeld in general)
Assume γ maps the generic point of D to $X^{nonsing}$
Then $\exists\, Y$ scheme of finite type & geometric point $y \in Y$ st $\widehat{\mathcal{L}(X)}_\gamma \simeq \underbrace{\widehat{\mathbb{A}}^\infty_0}_{\infty\text{-dim smooth part}} \times \widehat{Y}_y$ f.dim part

Can "speak of" IC sheaf $IC_{\mathcal{L}(X)}$ -- at least of its fibers: fiber of IC at γ as above is well defined. Moreover if k is finite the fiber is endowed with a Frobenius action [just IC sheaf of Y:
$\rightarrow$ (need to check independence of slice Y ---
char 0 obvious char p Gabber...)]

$\Rightarrow$ $Tr_{Fr}\, IC_{\mathcal{L}(X)}$ well defined function on $\mathcal{L}(X)^0 = \{\gamma \text{ as above}\}$

3. $G=T$. Embed $T \subset T_+$ normal algebraic affine semigroup with 1 s.t. T = group of invertible elements in T_+ [Vinberg]

e.g. $T = \mathbb{G}_m$, $T_+ = \mathbb{A}^1$

e.g. $\{xy = z^2\}$ semigroup $\supset T = \mathbb{G}_m \times \mathbb{G}_m$ (all three $\neq 0$)

– every normal toric variety locally looks like such T_+.

Look at arcs $T_+(\mathcal{O}) \cap T(K)$ $\quad K = k((t))$, $\mathcal{O} = k[[t]]$

i.e. arcs

$$
\begin{tikzcd}
D \arrow[r] & T_+ \\
D^* \arrow[u, phantom, "\cup"] \arrow[r] & T \arrow[u, phantom, "\cup"]
\end{tikzcd}
$$

$\Lambda = \mathrm{Hom}(\mathbb{G}_m, T)$ $\quad \Lambda^* = \mathrm{Hom}(T, \mathbb{G}_m)$

$k[T_+]$ spanned by subset of characters of T in that cone

$k[T_+] = \bigoplus_{\lambda^* \in \Lambda^*_+} k \cdot \lambda^*$, Λ^*_+ determines T_+,

Λ_+ dual cone (might have smaller dim! eg $T^+ = T \Rightarrow \Lambda_+ = 0$)

$T(\mathcal{O})$ orbits on $T(K) \longleftrightarrow \Lambda$ $\qquad T^\lambda \longleftrightarrow \lambda$

$T(K) \cap T_+(\mathcal{O}) = \bigcup_{\lambda \in \Lambda_+} T^\lambda$

$\gamma \in T^\lambda$ $\lambda \in \Lambda_+$ describe Y:

Λ_+ is a saturated semigroup in Λ, & $\Lambda_+ \cap -\Lambda_+ = 0$, f.g.

$\Rightarrow$ Λ_+ has unique minimal set of generators $\lambda_1 \ldots \lambda_n$

$\lambda \in \Lambda_+ \Rightarrow P(\lambda) = \{$decompositions $\lambda = \sum m_i \lambda_i\}$

<u>Theorem</u> $\gamma \in T^\lambda \Rightarrow$ can choose (Y, y) s.t. $Y_{red} = \bigcup$ smooth irred components, components $\xleftrightarrow{1:1} P(\lambda)$, dimension $= \sum m_i - 1$, intersect at $y \in Y$

Eg $xy = z^2$ $\quad \Lambda = \{(a,b,c) \mid a+b = 2c\}$

$(2,0,1)$ $\quad (0,2,1)$ $\quad (1,1,1)$ – generators

<u>Corollary</u> Let $\rho = \oplus \lambda_i$ rep of $T^\vee$ (λ_i cocharacters of T)

The function coming from IC of $\mathcal{L}(T_+) \cap T(K)$ is given by $\mathrm{Sym}^\bullet \rho$. i.e we've constructed $\mathcal{L}_\rho$ geometrically

($T(\mathcal{O})$-inv't function on $T_+(\mathcal{O}) \cap T(K)$ $\Longrightarrow$ rep of $T^\vee$)
– uniform description for all λ.

<u>General reductive G</u> Vinberg – reductive algebraic semigroups
$G \subset G_+$ G = group of invertible elements (open dense)
G_+ = normal affine & with unit

$I(_{1(G_+) \cap G(K)}$: corresponding function is $G(\mathcal{O})$-invariant on $G(K)/G(\mathcal{O}) \longleftrightarrow$ rep of $G^\vee$.

<u>Theorem</u> To G_+ one can associate a representation
$\rho : G^\vee \longrightarrow GL(n)$ s.t. above function corresponds to $\mathrm{Sym}\,\rho$ under Satake.
i.e. equal to $\mathcal{C}_\rho$
So can understand $\mathcal{C}_\rho$ for ρ coming from semigroups.
$\Rightarrow$ which ρ arise? all interesting ones:
e.g. any irrep of semisimple part of $G^\vee$ can be obtained

Example 0. $G = GL_n$, ρ = standard of $G^\vee = GL_n \Rightarrow G_+ = Mat(n)$
1. $G_+ = \{(g,t) \; g \in Mat(n), \; t \in \mathbb{A}^1, \det g = t^k\}$
for some <u>fixed</u> $k > 0$. $k=1 \Rightarrow$ matrices, $k > 1$: singular.
G is isogenous to $SL(n) \times \mathbb{G}_m$
$G^\vee$ = image of GL_n in $\underset{\rho}{\mathrm{Sym}^k V}$ V = standard.

2. $G_+ = \{ g \in Mat(n), h \in Mat(m) \; / \; \det g = \det h \}$
$\overset{\vee}{G}$: <u>nonzero</u> det . $G^\vee = GL_n \times GL_m / (\lambda Id_n, \lambda^{-1} Id_m)$

ρ = tensor product of standard reps
$\longrightarrow$ Rankin-Selberg L-function.

3. Vinberg's semigroup : G semisimple simply-conn,
take $\tilde{G} = G \underset{Z}{\times} T$, $G_+ \supset \tilde{G}$

$\tilde{G}^\vee = (G^\vee)^{\text{simply conn}} \underset{Z^\vee}{\times} (T^\vee)^{\text{simply conn}}$ $\rho = \bigoplus_i V(\omega_i)$
ω_i = fund weights

Q: 1. Does this make sense over $\mathbb{Q}_p$ instead of Laurent series?

2. What to do for ramified representations?

$S_\rho(G)$ for ρ coming from G_* ~~ functions coming by traces of Fr from perverse sheaves on $\mathcal{L}(G_*) \cap G(K)$

– in some examples can define this space of fns though not yet category of perverse sheaves

$S_\rho(G)^I = ?$ should have nice interpretation using equivariant K-theory of $G^\vee$

$S_\rho(G)^{G(\mathcal{O})}$ does have such nice explanation.

Note: for every toric variety have nice description of singularities of ~~jet~~ space ... what about nonrational singularities?

Note: a given ρ can correspond to at most one G_*.

$$K_{G^\vee}(V) \supset K_{G^\vee}(pt) = C_c^\infty(G(\mathcal{O})\backslash G(K)/G(\mathcal{O}))$$

- rationally an isomorphism (here V = space of ρ)

Iwahori: need to put V inside Steinberg...