Hitchin's connection, Higgs bundles and representations of the mapping class groups.

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- Σ closed oriented surface of genus $g \geq 2$.
- $p \in \Sigma$.
- P principal $G = SU(n)$ bundle over Σ
- $\xi \in Z_{SU(n)} \cong \mathbb{Z}/n\mathbb{Z}$
- M_ξ moduli space of flat G-connections in $P|_{\Sigma \setminus p}$ with holonomy ξ around p.
- $\langle , \rangle$ inv. inner product on $\mathfrak{g} = \mathrm{Lie}(G)$ s.t. $\langle \theta \wedge [\theta \wedge \theta] \rangle \in \mathrm{Im}(H^3(G,\mathbb{Z}) \to H^3(G,\mathbb{R}))$, where θ is the Maurer-Cartan form on G.

Facts about M:

- $M = \mathrm{Hom}_{\xi}(\pi_1(\Sigma \setminus p), G)/G$ compact.

- $M' =$ moduli space of irreducible flat G-connections in $P|_{\Sigma \setminus p}$ w. hol. ξ around p.

- M' smooth manifold of dimension $(2g-2)\dim G$.

  $[A] \in M'$: $\quad T_{[A]} M \cong H^1(\Sigma, \mathrm{ad})$

  $$\Omega^i(\Sigma, \mathrm{ad}P) \xrightarrow{d_A} \Omega^{i+1}(\Sigma, \mathrm{ad}P).$$

- If $(n, \xi) = 1$ then $M = M'$.

- M' symplectic

  $$\omega : T_{[A]} M \times T_{[A]} M \to \mathbb{R}$$

  $$\omega(\varphi_1, \varphi_2) = \int_{\Sigma} \langle \varphi_1 \wedge \varphi_2 \rangle$$

- M stratified symplectic space.

- $(\mathcal{L}, \nabla)$

  $\mathcal{L}$ Hermitian line bundle over M, smooth over M'

  ∇ Hermitian connection in $\mathcal{L}$ over M' s.t.

$$F_\nabla = \omega .$$

- Γ = mapping class group of Σ.

  Γ acts on $(M, \omega, \mathcal{L}, \nabla)$ by bundle transformations of $\mathcal{L}$ ~~with~~ which preserves ∇ and hence covers the symplectic action of Γ on M. ~~[illegible]~~

- $\mathcal{M}$ = moduli space of flat $G^{\mathbb{C}} = SL(n, \mathbb{C})$ - connections in $\mathcal{P} = P \times_G$ ~~[illegible]~~ $G^{\mathbb{C}}|_{\Sigma \setminus p}$ with holonomy $[\xi]$ around p.

- $M \subseteq \mathcal{M}$ and we have a compatible action of Γ on $\mathcal{M}$ by holomorphic transformation

- $\mathcal{T}$ = Teichmüller space of Σ.

 = moduli space of marked complex structures

 = a component of the moduli space of flat $SL(2,\mathbb{R})$ connections on Σ

- Notice $\mathcal{T} \hookrightarrow \mathcal{M}_{SL(2,\mathbb{C})}$.

- $\sigma \in \mathcal{T}$ gives a $*$- operator on 1-forms:

$$*: \Omega^1(\Sigma, \mathrm{ad}P) \circlearrowleft \qquad *^2 = -1.$$

Hodge theory:

$$H^1(\Sigma, \mathrm{ad}P) = \ker(d_A + *d_A*) = \mathcal{H}^1(\Sigma, \mathrm{ad}P)$$

Define a complex structure on $\mathcal{M}$ by

$$I : T_{[A]}\mathcal{M} \to T_{[A]}\mathcal{M}$$

$$\begin{array}{ccc} T_{[A]}\mathcal{M} & \longrightarrow & T_{[A]}\mathcal{M} \\ \cong & & \cong \\ \mathcal{H}^1(\Sigma, \mathrm{ad}P) & \xrightarrow{-*} & \mathcal{H}^1(\Sigma, \mathrm{ad}P) \end{array}$$

<u>Theorem</u> (Seshadri)

$M'_\sigma = (M', \omega, I)$ is a Kähler manifold.

<u>Theorem</u> (Narasimhan & Seshadri)

$M'_\sigma = (M', \omega, I) \cong$ moduli space of ~~stable~~-stable bundles of rank n and determinant isomorphic to $\xi \cdot [p]$, which is a ~~with~~ quasi projective variety with closure M_σ = moduli space of semi-stable .....

- $(\mathcal{L}, \nabla^{0,1})$ generates ample cone of $\mathrm{Pic}(M'_\sigma) \cong \mathrm{Pic}(M_\sigma)$.

- $H^0(M'_\sigma, \mathcal{L}^k) = H^0(M_\sigma, \mathcal{L}^k) \subseteq C^\infty(M', \mathcal{L}^k)$

Hitchin's Connection

- $\mathcal{T}$ = Teichmüller space of Σ.

- $\mathcal{H}$ = Trivial $C^\infty(M, \mathcal{L}^k)$ over $\mathcal{T}$

- $\mathcal{V} \subseteq \mathcal{H}$ Verlinde subbundle:

$$\mathcal{V}_\sigma = H^0(M_\sigma, \mathcal{L}^k).$$

- $T_\sigma \mathcal{T} = H^1(\Sigma_\sigma, K^{-1}).$

- $T_{[E]} M_\sigma = H^1(\Sigma_\sigma, \mathrm{End}_0(E))$

- $T^*_{[E]} M_\sigma = H^0(\Sigma_\sigma, \mathrm{End}_0(E) \otimes K)$

- $\dot{\sigma} \in H^1(\Sigma_\sigma, K^{-1}) = T_\sigma \mathcal{T}$

induces a $\dot{I} \in H^1(M, T)$:

$$I^2 = -1 \Rightarrow$$

$$\dot{I} I + I \dot{I} = 0 .$$

So $\dot{I}$ transforms $\begin{smallmatrix}-i\\ i\end{smallmatrix}$ eigenspace of I to $\begin{smallmatrix}+i\\ -i\end{smallmatrix}$ eigenspace, hence

$$\dot{I} \in \Omega^{0,1}(M, T)$$

Linearising the integrality condition for I

$$\bar{\partial} \dot{I} = 0 \in \Omega^{0,2}(M, T)$$

Hence

$$\dot{I} \in H^{0,1}(M, T) \cong H^1(M, T) .$$

Now $\omega \in \Omega^{1,1}(M)$. Define

$$G \in \Omega^0(M, S^2(T))$$

$$\dot{I} = G \cdot \omega \qquad \left(G = \dot{I}\,\omega^{-1}\right)$$

In fact $G \in H^0(M, S^2(T))$.

Explicit formula for G:

Think of $G \in H^0(M, S^2(T))$ as a quadratic function on T^*

$$T^*_E M = H^0(\Sigma_\sigma, \mathrm{End}_0(E) \otimes K)$$

$$G(\alpha, \alpha) = \int_\Sigma \mathrm{Tr}(\alpha \wedge \dot{*} \alpha)$$

$$= \int_\Sigma \mathrm{Tr}(\alpha^2) \dot{*} \qquad (*)$$

$(\dot{*} \in H^1(\Sigma_\sigma, K^{-1}) \cong H^{0,1}(\Sigma_\sigma, K^{-1}))$.

$\mathrm{Tr}(\alpha^2) \in H^0(\Sigma_\sigma, K^2) \cong T^*_\sigma \mathcal{T}$.

- $*_t$ curve in $\mathcal{T}$ with corresponding $\dot{*}$ and G for each t related by $(*)$.

· $s_t \in C^\infty(M, \mathcal{L}^k)$ is parallel w.r.t.
Hitchin's connection if

$$\dot{s} = u_G(s)$$

where

$$u_G(S) = \frac{1}{2k+\lambda}\left(\Delta_G - 2G\partial F\nabla + ik f_G\right)s$$

· Δ_G:

$$C^\infty(L^k) \xrightarrow{\nabla^{1,0}} C^\infty(T^*\otimes L^k) \xrightarrow{G} C^\infty(T\otimes L^k)$$

$$\xrightarrow{\nabla^{1,0}\otimes 1 + 1\otimes\nabla^{1,0}} C^\infty(T^*\otimes T\otimes L^k) \xrightarrow{Tr} C^\infty(L^k).$$

· F Ricci potential

$$\mathrm{Ric} = \lambda\omega - 2i\,\partial\bar{\partial}F$$

$$\partial F\otimes\nabla s \in C^\infty(T^*\otimes T^*\otimes \mathcal{L}^k)$$

• f_G:

$$\theta_G = 2i \ G \ \partial F \otimes \omega + \mathrm{Tr} \ \nabla^{1,0}(G\cdot\omega)$$

$$G\cdot\omega \in \Omega^{0,1}(T), \ \nabla^{1,0}(G\cdot\omega) \in \Omega^{1,1}(T)$$

$$\mathrm{Tr}(\nabla^{1,0}(G\cdot\omega)) \in \Omega^{0,1}.$$

$$\partial F \otimes \omega \in C^\infty(T^* \otimes T^* \otimes \bar{T}^*) = \Omega^{0,1}(T^* \otimes T^*)$$

then

$$\bar{\partial} f_G = \theta_G .$$

• u_G satisfies that

$$\boxed{\nabla^{0,1} u_G(S) = -i \dot{I} \nabla^{1,0} S}$$

which is the inf. condition for preserving holomorphicity

$$\nabla^{0,1} S = 0 \iff$$

$$(1 + i I) \nabla S = 0 \implies$$

$$i \dot{I} \nabla s + (1+iI)\nabla \dot{s} = 0 \iff$$

$$\nabla^{0,1} u_G(s) = -i \dot{I} \nabla^{1,0} s .$$

Axelrod Della Pietra & Witten /

<u>Theorem</u> (Hitchin / Faltings.)

This connection is projectively flat.

<u>Proof</u>

Let t_1 and t_2 be local hol. coordinates on Teichmüller space

$$\nabla_{t_1} = \frac{\partial}{\partial t_1} - u_{G_1}$$

$$\nabla_{t_2} = \frac{\partial}{\partial t_2} - u_{G_2}$$

and

$$[\nabla_{t_1}, \nabla_{t_2}] = \frac{\partial u_{G_1}}{\partial t_2} - \frac{\partial u_{G_2}}{\partial t_1} + [u_{G_1}, u_{G_2}]$$

Now u_{G_1} and u_{G_2} are 2^{nd} order operators, so is $\frac{\partial u_{G_1}}{\partial t_2}$ and $\frac{\partial u_{G_2}}{\partial t_1}$

and $[u_{G_1}, u_{G_2}]$ is 3^{rd} order apriori and

and

$$\sigma_3([\varkappa_{G_1}, \varkappa_{G_2}]) = \{\sigma_2(\varkappa_{G_1}), \sigma_2(\varkappa_{G_2})\}$$

$$= \{G_1, G_2\} = 0$$

Since G_i is part of Hitchin's integrable system.

Hence $[\nabla_{t_1}, \nabla_{t_2}] \in H^0(\mathcal{M}, D^2(\mathcal{L}^k))$ is at most 2^{nd} order.

Particular arguments show that $\sigma_2([\nabla_{t_1}, \nabla_{t_2}]) = 0$

But because $H^0(\mathcal{M}, T) = 0$, there are no non trivial 1-order holomorphic operators in $\mathcal{L}^k$, i.e. $\sigma_1([\nabla_{t_1}, \nabla_{t_2}]) = 0$.

Hence $[\nabla_{t_1}, \nabla_{t_2}]$ is just a constant c_{12} $\square$

From this connection we thus get a projective representation of the mapping class group

$$\rho_k : \Gamma \longrightarrow \mathrm{Aut}(PV_k)$$

where PV_k = cov. const. sections of $P(\mathcal{V})$
$$\downarrow$$
$$\mathcal{T}.$$

<u>Theorem</u> (A)

This sequence of representations ~~...~~ is asymptotic faithfull (for all $n \geq 2$):

$$\bigcap_{k \geq 1} \ker \rho_k = \{1\}$$

if $g \geq 2$.

<u>Proof</u>

Toeplitz operators: Let $f \in C_c^\infty(M')$ and consider

$$T_f^{(k)} : H^0(M'_\sigma, \mathcal{L}^k) \xrightarrow{f\cdot} C^\infty(M', \mathcal{L}^k) \xrightarrow{\pi} H^0(M'_\sigma, \mathcal{L}^k)$$

There is an induced flat connection in $\mathrm{End}(V)$ from Hitchin's proj. flat connection in V.

Theorem 1 (A)

$T_f^{(k)}$ is asymptotically flat as $k \to \infty$, i.e.

$$\| P_{\sigma',\sigma} T^{(k)}_{f,\sigma} - T^{(k)}_{f,\sigma'} \| \to 0 \quad \text{as } k \to \infty .$$

operator norm on $H^0(M, \mathcal{L}^k)$.

Suppose we now have a $\phi \in \Gamma$ s.t. $\phi \in \bigcap_{k \geq 1} \ker \rho_k^{m}$.

Then

$$\begin{tikzcd}
H^0(M_\sigma, \mathcal{L}^k) \arrow[r, "\phi^*"] \arrow[d, "T^{(k)}_{f,\sigma}"'] \arrow[rr, bend left, "\rho_k(\phi)"] & H^0(M_{\phi(\sigma)}, \mathcal{L}^k) \arrow[r, "P_{\sigma,\phi(\sigma)}"] \arrow[d, "T^{(k)}_{f\circ\phi,\phi(\sigma)}"] & H^0(M_\sigma, \mathcal{L}^k) \arrow[d, bend right, "\cancel{P}\ \alpha"'] \arrow[d, bend left, "T^{(k)}_{f,\sigma}"] \\
H^0(M_\sigma, \mathcal{L}^k) \arrow[r, "\phi^*"] \arrow[rr, bend right, "\rho_k(\phi)"'] & H^0(M_{\phi(\sigma)}, \mathcal{L}^k) \arrow[r, "P_{\sigma,\phi(\sigma)}"] & H^0(M_\sigma, \mathcal{L}^k)
\end{tikzcd}$$

$$\alpha = P_{\sigma,\phi(\sigma)} T^{(k)}_{f\circ\phi,\phi(\sigma)}$$

Since $f_k(\phi) \in \mathbb{C}\,\mathrm{Id}$ we get that

$$\| T^{(k)}_{f,\epsilon} - T^{(k)}_{f\circ\phi,\epsilon} \| \to 0 \quad \text{as } k \to \infty .$$

$$\|$$

$$T^{k}_{(f - f\circ\phi),\epsilon}$$

~~Berkeley~~ General theorem

<u>Theorem</u> (~~...~~ Schlichenmaier)

For $f \in C^\infty(M)$, M compact Kähler manifold w. $\begin{array}{c}\mathcal{L}\\ \downarrow \\ M\end{array}$ hol.

$$\| T_f \| \to \sup_{x \in M} |f(x)|$$

So we get that $f = f\circ\phi$ for all $f \in C^\infty_c(M)$

But this means that ϕ acts by identity on M_ξ for $u = 2$.

But ϕ acts holomorphically on $\mathcal{M}_\xi$ and $M_\xi \subseteq \mathcal{M}_\xi$ is real, i.e. $I_{\mathcal{M}_\xi}(TM_\xi) \perp TM_\xi$

But then ϕ acts by identity on the normal bundle to $M_\xi \subseteq \mathcal{M}_\xi$. Then ϕ must acts by identity in a nbhd of $\mathcal{M}_\xi$, hence by identity on all of $\mathcal{M}_\xi$, by the connectedness of $\mathcal{M}_\xi$.

But then ϕ acts by identity on $\mathfrak{J} \subseteq \mathcal{M}$ hence ϕ must be $1 \in \Gamma$.

Asymptotic covariant constant of $T_f^{(k)}$:

Let

$$A = \dot{T_f} - [u_G, T_f]$$

$$= \dot{\pi} f - [u_G, \pi f]$$

Claim 1

$$(1-\pi) A = 0$$

Proof

$$\pi = \pi^2 \Rightarrow \dot{\pi} = \pi \dot{\pi} + \dot{\pi} \pi \Rightarrow$$

$$(1-\pi)\dot{\pi} = (1-\pi)\dot{\pi}\pi$$

For all sections s s.t. $\dot{s} = u_G(s)$ $\pi s = s$ we get that

$$\dot{\pi} s + \pi \dot{s} = \dot{s} \Rightarrow$$

$$\dot{\pi} s = (1-\pi) u_G(s)$$

So $\dot{\pi}|_{H^0(M, L^k)} = (1-\pi) u_G|_{H^0(M, L^k)}$

$\square$

So therefore

$$\pi A \pi = A \pi : H^0(M, L^k) \longrightarrow H^0(M, L^k).$$

Claim 2

$$\lim_{k \to \infty} \| A \pi \|^{(k)} = 0$$

Proof.

A calculation shows that

$$\pi \dot{\pi} = \pi u_G^* - \pi u_G^* \pi$$

Recall that

$$u_G = \frac{1}{2k+\lambda} \left(\Delta_G - 2G\partial F \nabla + ik f_G \right)$$

So

$$u_G^* = \frac{1}{2k+\lambda} \left(\Delta_G^* - (2G\partial F \nabla)^* - ik \bar{f}_G \right)$$

So

$$\pi \dot{\pi} f \pi = \pi u_G^* f \pi - \pi u_G^* \pi f \pi$$

and

$$\pi [u_G, \pi f] \pi = \pi u_G \pi f \pi - \pi f u_G \pi$$

Using the symbol calculus for Toeplitz operators one checks that

$\pi A_G^* f \pi$, $\pi u_G^* \pi$, $\pi (2G\partial F\nabla)^* f \pi$, $\pi (2G\partial F\nabla^*)$

$\pi A_G \pi$, $\pi f A_G \pi$, $\pi\, 2G\partial F\nabla\, \pi$, $\pi f\, 2G\partial F\nabla\, \pi$

are zero's order ~~to~~ Toeplitz operators, hence all bounded in operator norm. This means that once scaled by $\frac{1}{2k+\lambda}$ these terms clearly go to zero in operator norm.

The result now follows from the fact that

$$\| \pi \bar{f}_G f \pi - \pi \bar{f}_G \pi f \pi \| = O\left(\frac{1}{k}\right)$$

and

$$\| \pi f_G \pi f \pi - \pi f f_G \pi \| = O\left(\frac{1}{k}\right)$$

since the zero' order symbol of these operator agree.

+ a bit more