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A counter-example to a 1961 theorem in homological algebra

Terminological conventions:

- an abelian category for us will always mean an abelian category with products (hence with all inverse limits)
- a Mittag-Leffler sequence in an abelian category is a sequence of epimorphisms

Given a Mittag-Leffler sequence

$$\twoheadrightarrow X_2 \twoheadrightarrow X_1 \twoheadrightarrow X_0$$

and a projective object P mapping to X_0 $\Rightarrow$ can lift $P \to X_0$ to a map $P \to X_1$ which in turn can be lifted to $P \to X_2$ etc.

This gives rise a map $P \longrightarrow \varprojlim X_i$ so that

$$\begin{tikzcd} P \arrow[r] \arrow[dr] & \varprojlim X_i \arrow[d] \\ & X_0 \end{tikzcd}$$

commutes.

Furthermore if $P \twoheadrightarrow X_0$ is a surjection $\Rightarrow \varprojlim X_i \twoheadrightarrow X_0$ is also a surjection

This gives

<u>Fact</u>: If $\mathfrak{A}$ - an abelian category with enough projectives $\Rightarrow \forall \{X_i\}$ M-L sequence we will have a surjection

$$\varprojlim X_i \twoheadrightarrow X_0 .$$

The natural question here is if this fact really depends on having enough projectives?

<u>Fact 2</u>: If $\mathfrak{A}$ - Sheaves on $\mathbb{R}^1$, then $\exists$ a ML sequence $\{X_i\}$ with $\varprojlim X_i = 0$

<u>Note</u>: The same ML sequence from Fact 2 has the property that

$$\prod_{i=0}^{\infty} X_i \longrightarrow \prod_{i=0}^{\infty} X_0$$

is not epi. So Maybe our original Fact has something to do with products.

<u>Def (Grothendieck)</u> An abelian category $\mathfrak{A}$ satisfies (AB4*) if any product of epis in $\mathfrak{A}$ is also an epi.

<u>Theorem (Jan-Erik Roos, 1961)</u> If $\mathfrak{A}$ is an abelian category satisfying (AB4*) $\Rightarrow$ $\forall$ ML sequence $\{X_i\}$ we get an epimorphism $\varprojlim X_i \twoheadrightarrow X_0$

We will show that this theorem is false.

<u>Example</u>: the ML sequence one uses in <u>Fact 2</u> is constructed as follows.

Let $a < b \in \mathbb{R}$, Let $j : (a,b) \hookrightarrow \mathbb{R}$ and put

$$F_{(a,b)} := j_! \mathbb{Z}$$

Take

$$X_i := F_{(0,(\frac{2}{3})^i)} \oplus F_{((\frac{1}{3})^i,1)} \twoheadrightarrow X_0 = F_{(0,1)}$$

Then $\varprojlim X_i = 0$ and $\prod X_i \to \prod X_0$ is not an epi.

To get the counter-example to the theorem we will need to construct some exotic abelian categories.

Let $\mathcal{J}$ - small additive category
Consider the category of all additive functors

$$\mathrm{Add}(\mathcal{J}^{op}, Ab)$$

This is a nice abelian category (has products, enough injectives, enough projectives, coproducts, etc.)

Let α be an infinite cardinal
Assume that $\mathcal{J}$ contains coproducts of $\leq \alpha$ objects.

Consider

$$E_X(\mathcal{J}^{op}, Ab) = \text{full subcategory of } \mathrm{Add}(\mathcal{J}^{op}, Ab) \text{ respecting the coproducts}$$

We will show that

- $E_X(\mathcal{J}^{op}, Ab)$ is abelian + has coproducts $\forall\ \mathcal{J}$ and every α
- the dual category of $E_X(\mathcal{J}^{op}, Ab)$ satisfies (AB4*) for a suitable choice

of $\mathcal{Y}$, α

• there is a sequence of monos in E_X which has limit $= 0$.

Remark: $E_X(\mathcal{Y}^{op}, Ab)$ has always coproducts. Indeed the natural inclusion

$$E_X(\mathcal{Y}^{op}, Ab) \hookrightarrow Add(\mathcal{Y}^{op}, Ab)$$

has a left adjoint

$$E_X(\mathcal{Y}^{op}, Ab) \leftrightarrows Add(\mathcal{Y}^{op}, Ab)$$

and since left adjoints preserve coproducts we are done

In order to show that $E_X(\mathcal{Y}^{op}, Ab)$ is abelian we can not use the left adjoint

$$E_X(\mathcal{Y}^{op}, Ab) \leftarrow Add(\mathcal{Y}^{op}, Ab)$$

since it is not exact.

Instead we will show that

$$E_X(\mathcal{Y}^{op}, Ab) \hookrightarrow Add(\mathcal{Y}^{op}, Ab)$$

embeds $E_X(\mathcal{Y}^{op}, Ab)$ as an abelian subcategory in $Add(\mathcal{Y}^{op}, Ab)$.

To show that we need to take any

$$F \to G \quad \text{- morphism in } E_\chi(\mathcal{Y}^{op}, Ab)$$

and show that if K, Q are its kernel and cokernel in $Add(\mathcal{Y}^{op}, Ab)$ $\Rightarrow$ K, Q are in $E_\chi(\mathcal{Y}^{op}, Ab)$.

To show this take any collection $\{X_\lambda, \lambda \in \Lambda\}$ of $\leq \alpha$ objects in $\mathcal{Y}$.

Since

$$0 \to K \to F \to G \to Q \to 0$$

is an exact sequence of presheaves on $\mathcal{Y}^{op}$ $\Rightarrow$ get an exact sequence

$$0 \to K(X_\lambda) \to F(X_\lambda) \to G(X_\lambda) \to Q(X_\lambda) \to 0$$

for all $\lambda \in \Lambda$.

$\Rightarrow$ get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & K(\coprod X_\lambda) \arrow[r] \arrow[d] & F(\coprod X_\lambda) \arrow[r] \arrow[d, "\wr"] & G(\coprod X_\lambda) \arrow[r] \arrow[d, "\wr"] & Q(\coprod X_\lambda) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \prod K(X_\lambda) \arrow[r] & \prod F(X_\lambda) \arrow[r] & \prod G(X_\lambda) \arrow[r] & \prod Q(X_\lambda) \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow$ the two maps at the ends are also $\cong$.

Now we need to find the right $\mathcal{Y}$ and α so that we will get a counter-example to Roos' theorem.

<u>Fact 3</u>: Representable functors

$$\mathcal{Y}(-,s) : \mathcal{Y}^{op} \to Ab$$

lie in $E_x(\mathcal{Y}^{op}, Ab)$. In particular $E_x(\mathcal{Y}^{op}, Ab)$ is not empty.

<u>Fact 4</u>: The Yoneda map

$$Y : \mathcal{Y} \to E_x(\mathcal{Y}^{op}, Ab)$$

(from Fact 3) respects coproducts of $\le \alpha$ objects.

<u>Proposition</u>: If weak kernels exist and coproducts of $\le \alpha$ weak kernels is a weak kernel, then $E_x(\mathcal{Y}^{op}, Ab)$ satisfies [AB4]

There are by now two choices of α and $\mathcal{Y}$ which give counterexamples to Roos' theorem (these are due to Neeman and Deligne).

For example (Neeman) one can take:

$$\alpha = \aleph_0 \text{ - the countable cardinal}$$

$$Ob(\mathcal{Y}) = \text{complete non-archimedian topological abelian groups of cardinality} \le 2^{\aleph_0}$$

$$Mor(\mathcal{Y}) = f : A \to B \quad \text{s.t.} \quad \|f(a)\| \le \|a\|$$

The example above naturally occurs when one studies triangulated categories.

Namely: given $\mathcal{T}$ - triangulated category we can study $\mathcal{T}$ by taking $\mathcal{Y} \subset \mathcal{T}$ - additive subcategory and looking at

$$\mathcal{T} \xrightarrow{Y} Ex(\mathcal{Y}^{op}, Ab)$$

One can check that $Ex(\mathcal{Y}^{op}, Ab)$

satisfies (AB4) but in general do not have enough injectives. The really important question is if $\lim^1$ vanishes on ML sequences in $E_X(\mathcal{D}^{op}, Ab)$.