EXAMPLES OF MIN. AFFINISATIONS

1. Drinfeld's Yangian

$\mathfrak{g}$ simple Lie alg / $\mathbb{C}$

$\mathfrak{g} \neq \mathfrak{sl}_2$

$$[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

$$X = \ker [\,,\,] \subseteq \Lambda^2 \mathfrak{g}$$

lemma: $\mathrm{Hom}_{\mathrm{Aut}\,\mathfrak{g}}(X, \mathcal{U}_{\leq 3}\mathfrak{g}) \xleftarrow{\sim} \underbrace{(S^2\mathfrak{g}^*)^{\mathfrak{g}}}_{\text{traces}} \cong \mathbb{C}$

$\hookleftarrow$ universal formula;

$$x \wedge y \mapsto \sum_{\alpha} ([[x I_\alpha][y I_\beta], I_\gamma)\,\{I_\alpha I_\beta I_\gamma\}$$

$(\,,\,)$ trace; $\{abc\} = \frac{1}{6}\sum_{\sigma \in S_3} \sigma(abc)$, I_α orb of $\mathfrak{g}$ wrt $(\,,\,)$

def: $Y(\mathfrak{g})$ Drinfeld's Yangian. (Hopf) alg deform of $\mathcal{U}\mathfrak{g}[t]$

generators: $\mathfrak{g} + \mathfrak{g}$

denote this copy $J(\mathfrak{g})$)

relations (i) $\mathfrak{g}$ Lie alg; so map $\mathcal{U}\mathfrak{g} \to Y(\mathfrak{g})$

(ii) $J(\mathfrak{g})$ adjoint rep of $\mathfrak{g}$

(iii) map

$$\begin{array}{ccc} \Lambda^2\mathfrak{g} & \longrightarrow & Y(\mathfrak{g}) \\ \cup & & \uparrow \\ X & \longrightarrow & \mathcal{U}_{\leq 3}\mathfrak{g} \end{array}$$

$x \wedge y \mapsto [Jx, Jy]$

factors

in an $\mathrm{Aut}\,\mathfrak{g}$ equi way.

(no other relations if $\mathfrak{g} \neq \mathfrak{sl}_2$; otherwise some).

[illegible]

def: $\mathbb{C} \longrightarrow \mathrm{Aut}\, V(\mathfrak{g})$ $\quad \gamma \mapsto t_\gamma$

$$t_\gamma x = x \qquad x \in \mathfrak{g}$$

$$t_\gamma J(x) = J(x) + \gamma x$$

outer aut.

§4 Drinfeld Polynomials

PROBLEM: describe cat of $\underline{\underline{f.d}}$ reps of $Y(\mathfrak{g})$

In particular, parametrise reps & ~~describe~~ compute characters / dimensions

· X wt lattice of $\mathfrak{g}$, X^+ dom wts, $L_\lambda \overset{= L_\lambda^{\mathfrak{g}}}{}$ irred f.d rep wt λ.

$\lambda = \sum n_i \omega_i \in X^+$, Write $S^\lambda \mathbb{C} = \prod S^{n_i}\mathbb{C}$ $\quad = \operatorname{Spec} H_{GL_\lambda}(\cdot)$

$(= \mathbb{C}^{n_i}/S_{n_i} = n_i$ unordered pts in $\mathbb{C}$)

thm (Drinfeld) $\left(\left(\text{Irred reps of } Y(\mathfrak{g})\right)\right) \longleftrightarrow \coprod_{\lambda \in X^+} S^\lambda \mathbb{C}$

$\qquad V(P_\lambda) \qquad \longleftrightarrow \qquad P_\lambda = (\lambda, \overset{\psi}{x})$ "Drinfeld poly"

ie: ~~irr~~ f.d reps have moduli $\qquad \lambda$ discrete param, + cts param $\in S$

① →

Moreover, if V irred rep ~~wt of~~ of $Y(\mathfrak{g})$ with discrete param λ,

$$V|\mathfrak{g} = L_\lambda + \sum_{\mu < \lambda} a_{\mu\lambda} L_\mu$$

$\mu \leq \lambda \Leftrightarrow \lambda - \mu \in \mathbb{N}\Phi^+$

① t_x acts ~~by~~ in obvious way: (simultaneous translation)

in particular; ~~faithfully~~ freely on ((Irred Reps))

rmk: $\operatorname{Rep}^\gamma(\mathfrak{g})$ has general structure of "$KL^{/BB}$ type":

stds, irred, ... Ext vanishing

which are apparent from geometry; but much richer structure.

which explains why can compute with sean.

$\hbar$ KR modules

prop [Drinfeld, Jimbo]
exists an alg homo $Y(\mathfrak{gl}_n) \xrightarrow{ev} U(\mathfrak{gl}_n)$

so: $\lambda \in X^+$, $ev\, L_\lambda = P(\lambda, \lambda(\omega))$
nice discrete set of reps in $S^\lambda \mathbb{C}/\mathbb{C}$
small, rigid reps

Q: what is analogue for other gps?

note: homo $Y(\mathfrak{g}) \to U\mathfrak{g}$ must factor $J(\mathfrak{g})$ $J: \mathfrak{g} \to U^{\leq 2}\mathfrak{g}$ $\xrightarrow{\ne 0} S^2\mathfrak{g}$

but $\mathrm{Hom}_{\mathfrak{g}}(\otimes \mathfrak{g}, S^2\mathfrak{g}) = 0$, $\mathfrak{g} \neq \mathfrak{sl}_n$

alternate, remarkable characterisation of certain λ eval. reps of $\mathfrak{sl}_n$, $\lambda = n\omega_i$
from TBA, due to Kirillov - Reshetikhin

KR-conjecture: $\exists$ irred $Y(\mathfrak{g})$ reps $KR_{n\omega_i}$ $n \in \mathbb{N}$
s.t. non-split exact sequence

Q-system

$$0 \to \underbrace{\bigotimes_{j: a_{ij}=-1} KR_{n\omega_j}}_{\text{irred}} \to t_{-h} KR_{n\omega_i} \otimes t_h KR_{n\omega_i} \to \underbrace{t_{-h} KR_{(n+1)\omega_i} \otimes t_h KR_{(n-1)\omega_i}}_{\text{irred}}$$

where: $\otimes$ uses Hopf alg structure,
h determined by multiple $X \hookrightarrow U_{\mathfrak{sl}_2}\mathfrak{g}$ of std embedding
$\mathfrak{g}$ simply laced in above (for simplicity)

remarks / ~~on~~ thms ~~for~~ (KR, others) Kuhn

(i) evaluation reps for $Y(\mathfrak{gl}_n)$ of this form

~~(ii)~~ $\underline{\underline{\text{Not}}}$ obvious

(ii) true for A, D, some E [Chari, Chari-Kleber]

(iii) this is an $\underline{\underline{\text{over}}}$ determined system of eqns

in particular, (*) determines $KR_{n\omega_i}\Big|_{q}$

(Kuniba, Nakanishi, Tsuboi, Okado Schilling, Shimozono, Kleber, Chari,)

(iv) Despite (iii) structure of $KR_{n\omega_i}$ $\underline{\underline{\text{not}}}$ clear from these eqns. Requires much combinatorial work

so. will define KR_λ for classical alg with structure as clear as eval reps.

illuminates ~~known~~ all above beautiful work.

v) $KR_{n\omega_i}$ related to Demazure modules.

$\mathfrak{f}$ Construction

EXAMPLE: $Y(\mathfrak{o}_n)$

$W = L + L^*$ symplectic v.s., L, L^* lagrangian; V quad. v.s

$V \otimes W$ symp. v.s

construct functor $F: \mathfrak{gl}_W\text{-mods} \to Y(\mathfrak{o}_V)\text{-mods}$

ie: construct then B a $Y(\mathfrak{o})$-$\mathfrak{gl}_W$ bimodule;

$$F(V) = \mathrm{Hom}_{\mathfrak{gl}_W}(B, \cdot)$$

def of B:

let $\mathcal{F} = \mathrm{Sym}(V \otimes L)$ = Fock space for $\mathrm{Heis}_{V\otimes L}$

$$0 \to \mathbb{C} \to \mathrm{Heis}_{V\otimes L} \to V \otimes W \to 0$$

$\mathfrak{sp}(V \otimes W)$ acts on $\mathcal{F}$

$\mathfrak{o}_V \times \mathfrak{sp}_W$ "dual pair": full commutants of each other in $\mathrm{End}\,\mathcal{F}$

$\mathcal{U}\mathfrak{o}_V \twoheadrightarrow \mathrm{End}_{\mathfrak{sp}(W)} \mathcal{F}$ surjects.

~~key lemma: $Y(\mathfrak{o})$~~

put $B = \mathrm{Ind}_{\mathfrak{sp}(W)}^{\mathfrak{sl}(W)} \mathcal{F} = \mathcal{U}\mathfrak{sl}_W \otimes_{\mathcal{U}\mathfrak{sp}_W} \mathcal{F}$

key lemma: $Y(\mathfrak{o}) \twoheadrightarrow \mathrm{End}_{\mathfrak{sl}_W}(B)$

$\boxed{k = \mathfrak{sp}_W}$

omit pf, but construct map:

$$\mathrm{End}_{\mathfrak{sl}_W}(B) = \mathrm{Hom}_{\mathfrak{sl}_W}(\mathrm{Ind}\,\mathcal{F}, \mathrm{Ind}\,\mathcal{F}) = \mathrm{Hom}_{\mathfrak{sp}(W)}(\mathcal{F}, \mathrm{Ind}\,\mathcal{F})$$

so filtered,

$$\mathrm{gr}\,\mathrm{End}(B) = \left(\mathrm{Sym}(\mathfrak{sl}_W/\mathfrak{sp}_W) \otimes \mathrm{End}\,\mathcal{F}\right)^k$$

$\mathcal{U}\mathfrak{o}_V \twoheadrightarrow \mathrm{End}\,\mathcal{F} \xhookrightarrow{\mathrm{Ind}} \mathrm{End}\,B$ / so enough to construct

ie $1 \otimes \mathrm{End}_k \mathcal{F}$ subalg

$$\begin{tikzcd}
J: \mathfrak{o}_V \arrow[r] \arrow[dr, "\text{lift}"'] & \left[\mathfrak{sl}_W/\mathfrak{sp}_W \otimes \mathrm{End}(\mathcal{F})\right]^K \\
& \left(\mathfrak{sl}_W/\mathfrak{sp}_W \otimes \mathfrak{sp}(V\otimes W)\right)^K \arrow[u]
\end{tikzcd}$$

but $\mathfrak{sp}(V\otimes W) \simeq S^2(V\otimes W)$ as rep of $\mathfrak{sp}(V\otimes W)$, hence

$\simeq S^2V \otimes S^2W + \Lambda^2V \otimes \Lambda^2W$ ~~as~~

$\simeq \mathfrak{gl}_V/\mathfrak{o}_V \otimes \mathfrak{sp}(W) + \mathfrak{o}_V \otimes \mathfrak{gl}_W/\mathfrak{sp}_W$

as rep of $\mathfrak{o}_V \times \mathfrak{sp}(W)$

so $J: \mathfrak{o}_V \to \mathfrak{sl}_W/\mathfrak{sp}_W \otimes \mathfrak{o}_V \otimes \mathfrak{gl}_W/\mathfrak{sp}_W$

natural trace map is $\mathfrak{o}_V \times \mathfrak{sp}_W$ invariant.

claim: (i) satisfies defining relation of $Y(\mathfrak{o}_V)$

(ii) surjects

omit.

call irred f.d $Y(\mathfrak{g})$-mods in image "gen. KR modules"

Immediate cor: structure of gen KR-modules as $\mathfrak{g}$-modules

roughly: if L_λ^{gl} $\hat{gl}_W$-rep, $F(L_\lambda) =: KR_\lambda$ irred $Y(\mathfrak{g})$-mod

& $\mathrm{Res}^{gl_W}_{sp_W} L_\lambda^{gl} = \sum a_{\mu\lambda} L_\mu^{sp}$ $\boxed{\text{known \& easy}}$ Littlewood

then $\mathrm{Res}^{Y(\mathfrak{g}_V)}_{\mathfrak{g}_V} KR_\lambda = \sum a_{\mu\lambda} L_\mu^{\mathfrak{g}}$

pf: Frobenius reciprocity.

precise version: identify (i) param set λ (ii) cat of modules
stable & unstable versions.

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Unstable version: def

~~unitarygration~~

cat of $\left\{ \begin{array}{l} (gl_W, gl_L \times gl_{L^*})\text{-mods} \\ \text{unitary}, \ \& \ (L^*)^{\otimes 2} \text{ acts locally nil'p} \end{array} \right\} =:$ Unitary h.w gl_W ~~gl~~

Res $\downarrow$

cat of $\left\{ \begin{array}{l} (sp_W, gl_L)\text{-mods} \\ \text{unitary} \ \& \ S^2 L^* \text{ acts locally nil'p} \end{array} \right\} =:$ unitary sp_W h.w mods

|| "hol. dis"

clear: (i) Restriction defined; Res (simple) = s.s.

(ii) L f.d $\mathfrak{g}_V$-mod $\Rightarrow$ $\mathrm{Hom}_{\mathfrak{g}_V}(\mathcal{F}, L) \in$ unitary h.w sp_W-mod.

Now: $\mathrm{Hom}_{gl_W}(B, -)$: (unitary gl_W-mods) $\longrightarrow$ f.d $Y(\mathfrak{g}_V)$-mods

Stable version:

can replace above cats with
$\underline{\underline{\text{fin'd}}}$ $\mathfrak{gl}_W$ modules in (∞ $\mathfrak{gl}_\infty$ / $\mathfrak{sp}_\infty$).

& describe pairs.

cor: these modules are those predicted by
KR, Kleber, H ..... ~.

Rmk