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NONABELIAN GAUGE THEORY ON A NONCOMMUTATIVE SPACE

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RAYMOND STORA & ALAN WEINSTEIN

- SEIBERG-WITTEN MAP (S-WM) AND ITS RELATION TO SUPERSTRING THEORY WITH CONSTANT B_{ij} FIELD.
- MUNICH GROUP REFORMULATION
- OUR REFORMULATION WITH GHOSTS: COHOMOLOGICAL METHOD
- BARNICH, BRANDT, GRIGORIEV, HENNEAUX: FORMULATION WITH B.V. ANTIFIELDS
- BARS. DOUGLAS, LIU, MOORE, ZWIEBACH: CAN MOYAL'S $*$ PRODUCT BE GENERALIZED TO WITTEN'S IN OPEN STRING FIELD THORY ?

WEYL-MOYAL FORMULA

$$[x^k, x^l] = i\,\theta^{kl} \qquad \theta^{kl} = -\theta^{lk}$$

$$f_c(x) \to \hat{f} \qquad g_c(x) \to \hat{g}$$

$$h_c(x) \to \hat{h} = \hat{f} \cdot \hat{g}$$

TOTAL SYMMETRIZATION

$$h_c(x) = f_c(x) * g_c(x) =$$

$$= e^{\frac{i}{2}\frac{\partial}{\partial x^k}\theta^{kl}\frac{\partial}{\partial y^l}} f_c(x)\, g_c(y)\Big|_{y\to x} =$$

$$= f_c(x)\, g_c(x) + \frac{i}{2}\theta^{kl}\frac{\partial f_c}{\partial x^k}\frac{\partial g_c}{\partial x^l} + [\theta^2]$$

STAR PRODUCT IS ASSOCIATIVE

$$\big(f_c(x) * g_c(x)\big) * K_c(x) = f_c(x) * \big(g_c(x) * K_c(x)\big)$$

BUT NOT COMMUTATIVE.

θ^{kl} INDEPENDENT OF x

$$\partial_i (f_c * g_c) = \partial_i f_c * g_c + f_c * \partial_i g_c$$

SEIBERG-WITTEN MAP

COMMUTATIVE THEORY:

GAUGE PARAMETER α, GAUGE POTENTIAL a_i, GAUGE TRANSFORM.

$$\delta_\alpha a_i = \partial_i \alpha - i[a_i, \alpha]$$

NONCOMMUTATIVE THEORY:

GAUGE PARAMETER

$$\Lambda = \Lambda(\alpha, \partial\alpha, \dots, a, \partial a, \dots),$$

GAUGE POTENTIAL

$$A_i = A_i(a, \partial a, \partial^2 a, \dots),$$

GAUGE TRANSFORM.

$$\delta_\Lambda A_i = \partial_i \Lambda - i(A_i * \Lambda - \Lambda * A_i).$$

REQUIRE SIMULTANEOUS

$$\boxed{A_i + \delta_\Lambda A_i = A_i(a_j + \delta_\alpha a_j, \dots)}$$

EQUATION FOR BOTH A_i AND Λ

MUNICH FORMULATION

$$\delta_\alpha \psi^{(0)} = i\,\alpha\,\psi^{(0)}$$

$$(\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha)\psi^{(0)} = [\alpha, \beta]\,\psi^{(0)} = -i\delta_{[\alpha,\beta]}\,\psi^{(0)}$$

$$\psi = \psi^{(0)} + \psi^{(1)} + \cdots = \psi(\psi^{(0)}, a_i)$$

$$\delta_\alpha \psi = i\,\Lambda_\alpha * \psi \qquad \delta_\beta \delta_\alpha \psi = i\,\Lambda_\alpha * \underbrace{\delta_\beta \psi}_{i\Lambda_\beta * \psi}$$

$$+\, i\,\delta_\beta \Lambda_\alpha\, \psi$$

$$[\delta_\beta, \delta_\alpha]\psi = i(\delta_\beta \Lambda_\alpha - \delta_\alpha \Lambda_\beta) * \psi +$$

$$+\,(\Lambda_\beta * \Lambda_\alpha - \Lambda_\alpha * \Lambda_\beta) * \psi$$

REQUIRE

$$-i\,\delta_{[\alpha,\beta]}\psi = -i \cdot i\,\Lambda_{[\alpha,\beta]} * \psi$$

$$\boxed{\delta_\beta \Lambda_\alpha - \delta_\alpha \Lambda_\beta - i\,\Lambda_{[\beta,\alpha]} = i(\Lambda_\beta * \Lambda_\alpha - \Lambda_\alpha * \Lambda_\beta)}$$

STRUCTURE EQ.S OF A GAUGE GROUP.

REPLACE THE GAUGE PARAMETER $\alpha(x)$ BY A GHOST FIELD $\lambda(x)$, THE INFINITES. GAUGE TRANSF. δ_α BY THE BRST OPERATOR s

$a_i(x)$

$$\begin{cases} s\lambda = i\lambda\lambda \\ s a_i = \partial_i \lambda - i a_i \lambda + i \lambda a_i \end{cases} \qquad s\partial_i = \partial_i s$$

GRADING: λ ODD, GHOST # 1
a_i EVEN, " 0

s IS AN ODD SUPERDERIVATION OF GHOST # 1. $s(fg) = sf\, g \pm f\, sg$, $s^2 = 0$

PROBLEM: DEFORM THE ABOVE STRUCTURE EQ.S INTO

$$\begin{cases} s\Lambda = i\Lambda * \Lambda \\ sA_i = \partial_i \Lambda - i A_i * \Lambda + i \Lambda * A_i \end{cases} \qquad s^2 = 0$$

$\Lambda = \Lambda(\lambda, \partial\lambda, \cdots, a, \partial a, \cdots)$ ODD, GHOST # 1

$A_i = A_i(a, \partial a, \partial^2 a, \cdots)$ EVEN, GHOST # 0

"LOCAL" FUNCTIONALS. s UNDEFORMED.

EXPANSION IN θ^{ij}

$$\Lambda = \Lambda^{(0)} + \Lambda^{(1)} + \cdots$$

$$\Lambda^{(0)} = \lambda \, , \quad \Lambda^{(1)} = \frac{1}{4}\theta^{kl}\{\partial_k\lambda, a_l\}, \cdots$$

$$A_i = A_i^{(0)} + A_i^{(1)} + \cdots$$

$$A_i^{(0)} = a_i \quad , \quad A_i^{(1)} = -\frac{1}{4}\theta^{kl}\{a_k, \partial_l a_i + F_{li}\}, \cdots$$

where $F_{li} = \partial_l a_i - \partial_i a_l - i[a_l, a_i]$.

EXPRESSIONS FOR $\Lambda^{(2)}$ AND $A_i^{(2)}$ ARE KNOWN

$$\Lambda = \Lambda^{(0)} + \Lambda^{(1)} + \Lambda^{(2)} + \cdots$$

$$\Lambda^{(0)} = \lambda \qquad \Lambda^{(1)} = \frac{1}{4}\theta^{kl}\{\partial_k\lambda, a_l\}$$

$$\Lambda^{(2)} = \frac{1}{32}\theta^{ij}\theta^{kl} \cdot$$

(MUNICH, hep-th/0104153)

$$\cdot\Big(-i\{\partial_i\lambda, \{a_k, [a_j, a_l]\}\}$$

$$-4\{\partial_i\lambda, \{a_k, \partial_l a_j\}\}$$

$$-i\{a_j, \{a_l, [\partial_i\lambda, a_k]\}\}$$

$$+2i[\partial_i\partial_k\lambda, \partial_j a_l]$$

$$-2[\partial_j a_l, [\partial_i\lambda, a_k]]$$

$$+2i[[a_j, a_l], [\partial_i\lambda, a_k]]\Big)$$

(GOTO & HATA, hep-th/0005101 v2)

BASIC EQS. $s(f*g) = sf*g \pm f*sg$, $[s, \partial_i] = \ldots$

$$s\Lambda = i\Lambda * \Lambda \qquad sA_i = \partial_i\Lambda - i[A_i \overset{*}{,} \Lambda]$$

INTRODUCE A PARAMETER t

$$\Lambda \to \Lambda(t) \qquad A_i \to A_i(t) \qquad \theta \to t\theta$$

s INDEPENDENT OF t

DIFFERENTIATE

$$s\dot\Lambda = i\dot\Lambda * \Lambda + i\Lambda * \dot\Lambda + i\Lambda \,\dot{*}\, \Lambda$$

$$* = e^{\frac{i}{2}t\partial\theta\partial} \qquad \dot{*} = e^{\frac{i}{2}t\partial\theta\partial}\,\frac{i}{2}\partial\theta\partial$$

$$f \,\dot{*}\, g = i\frac{\theta^{kl}}{2}\partial_k f * \partial_l g$$

$$\Delta\dot\Lambda \equiv s\dot\Lambda - i\Lambda*\dot\Lambda - i\dot\Lambda*\Lambda = -\frac{\theta^{kl}}{2}\partial_k\Lambda * \partial_l\Lambda$$

SIMILARLY

$$\Delta\dot A \equiv s\dot A_i - i[\Lambda \overset{*}{,} \dot A_i] = D_i\dot\Lambda +$$

$$+\frac{1}{2}\theta^{kl}\left(\partial_k A_i * \partial_l\Lambda - \partial_k\Lambda * \partial_l A_i\right)$$

$$D_i \equiv \partial_i - i[A_i \overset{*}{,} \cdot\,]$$

$$\Delta^2 = 0 \qquad [D_i, \Delta] = 0$$

ONE IS LED NATURALLY TO INTRODUCE THE OPERATORS

$$\Delta \cdot = s \cdot - i[\lambda \overset{*}{,} \cdot] \qquad \cdot \ \text{EVEN}$$

$$\Delta \cdot = s \cdot - i\{\lambda \overset{*}{,} \cdot\} \qquad \cdot \ \text{ODD}$$

AND

$$D_i \cdot = \partial_i \cdot - i[a_i \overset{*}{,} \cdot] \, .$$

Δ IS A (SUPER-) DERIVATION, LIKE S, AND SATIFIES

$$\Delta^2 = 0 \qquad \Delta D_i = D_i \Delta \, .$$

THESE EQ.S FOLLOW FROM

$$s^2 = 0 \qquad s\partial_i = \partial_i s$$

AND THE ASSOCIATIVITY OF $*$.

SIMILARLY $\Delta(f * g) = \Delta f * g \pm f * \Delta g$

FROM $s(f * g) = sf * g \pm f * sg$

CONVENIENT ABBREVIATION $B_k \equiv \partial_k \Lambda$

$$\Delta \dot{\Lambda} = -\frac{1}{2}\theta^{kl} B_k * B_l$$

$$\Delta \dot{A}_i = D_i \dot{\Lambda} + \frac{1}{2}\theta^{kl}\{\partial_k A_i \overset{*}{,} B_l\}$$

SOLUTION: THE EVOLUTION EQ.S

$$\dot{\Lambda} = \frac{1}{4}\theta^{kl}\{B_k \overset{*}{,} A_l\}$$

$$\dot{A}_i = -\frac{1}{4}\theta^{kl}\{A_k \overset{*}{,} \partial_l A_i + F_{li}\}$$

$$F_{li} = \partial_l A_i - \partial_i A_l - i[A_l \overset{*}{,} A_i]$$

$$\Delta B_k = 0 \qquad \Delta A_k = B_k \qquad \Delta^2 = 0$$

Δ IS NOT INVERTIBLE

HOMOTOPY OPERATOR K

$$K B_k = A_k \qquad K A_k = 0$$

$K\Delta + \Delta K = 1$ ON BOTH B & A

Δ IS A (SUPER-) DERIVATION, $\Delta^2 = 0$

Δ IS NOT INVERTIBLE

$\Delta B_k = 0 \qquad \Delta A_k = B_k$.

INTRODUCE THE HOMOTOPY OPERATOR

$\tilde{K} B_k = A_k \qquad \tilde{K} A_k = 0$.

ON BOTH B_k & A_k

$$\tilde{K}\Delta + \Delta\tilde{K} = 1$$

BUT $\tilde{K}$ CANNOT BE A (S-)DERIVATION

ON MONOMIALS IN B & A

$K = \frac{1}{D}\tilde{K} \qquad D =$ DEGREE OF MONOM.

THEN $K\Delta + \Delta K = 1 \qquad K^2 = 0$.

IF $\Delta b = m$, $\Delta m = 0$, then

$m = (K\Delta + \Delta K)m = \Delta K m \qquad b = Km + \Delta n$

ASAKAWA & KISHIMOTO AMBIGUITY

$s\Lambda = i\Lambda * \Lambda$ CHANGE θ BY $\delta\theta$

$$s\delta\Lambda = i\,\delta\Lambda * \Lambda + i\Lambda * \delta\Lambda - \frac{1}{2}\delta\theta^{kl}\partial_k\Lambda * \partial_l\Lambda$$

$$\Delta\delta\Lambda = -\frac{1}{2}\delta\theta^{kl}\partial_k\Lambda * \partial_l\Lambda$$

GIVEN A SOLUTION $(\delta\Lambda)_0$ OF THIS EQ.
$(\delta\Lambda)_0 + \Delta H$ IS ALSO A SOLUTION $(\Delta^2 = 0)$

SIMILARLY FROM
$sA_i = \partial_i\Lambda - i[A_i \overset{*}{,} \Lambda]$ ONE FINDS

$$\Delta\,\delta A_i - D_i\,\delta\Lambda = \frac{1}{2}\delta\theta^{kl}\{\partial_k A \overset{*}{,} \partial_l\Lambda\}$$

GIVEN A SOLUTION $(\delta A_i)_0$ (WITH $(\delta\Lambda)_0$ ABOVE)
$(\delta A_i)_0 + D_i H + S_i$ (WITH $(\delta\Lambda)_0 + \hat{\Delta} H$ & $\Delta S_i = 0$)
IS ALSO A SOLUTION $(D_i\Delta = \Delta D_i)$

$\Delta S_i = 0$ MEANS $sS_i = i\Lambda * S_i - iS_i * \Lambda$ i.e.
S_i IS COVARIANT. $\delta A_i = S_i$ IS A
REDEFINITION OF THE GAUGE POTENTIA

A-K AMBIGUITY IS AN INFINITESIMAL VERSION OF THE STORA INVARIANCE.

THE BASIC EQ.S

$$s\Lambda = i\,\Lambda * \Lambda \qquad sA_i = \partial_i \Lambda - i\,[A_i \,\overset{*}{,}\, \Lambda]$$

ARE COVARIANT UNDER THE TRANSF.

$$\Lambda \to \Lambda' = G^{-1} * \Lambda * G + i\, G^{-1} * s\, G$$

$$A_i \to A_i' = G^{-1} * A_i * G + i\, G^{-1} * \partial_i G$$

WHERE $G = G(a, \partial a, \partial^2 a, \dots)$ IS AN ARBITRARY LOCAL FUNCTIONAL OF GHOST # 0