# Minimal Surfaces of Rotation in a special Randers Space

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# Special Randers Space

$(V^{n+1}, F_b)$, $(n+1)$-dimensional real vector space equipped with a Randers metric

$$F_b(x, y) = \alpha(x, y) + \beta(x, y),$$

$(x, y)$ is in the tangent bundle TV,

$\alpha$ is the Euclidean metric,

$\beta$ is a 1-form whose norm b satisfies $0 \leq b < 1$.

w.l.o.g $\beta = b\, dx_{n+1}$.

We will consider:

- Differential equation for a minimal immersion $\varphi : M^n \to (V^{n+1}, F_b)$
- ODE for minimal surfaces of rotation in $(V^3, F_b)$ with rotating axis $x_3$.

## Theorem

Up to homothety, there exists a unique forward complete minimal surface of rotation on a special Randers space $(V^3, F_b)$, for each b, $0 \leq b < 1$.

- The surface is embedded;
- Symmetric with respect to a plane perpendicular to the rotation axis;
- It is generated by a concave plane curve.
- When $\sqrt{3}/3 < b < 1$, the slope of the tangent lines to the generating curve is bounded by

$$\pm\frac{\sqrt{1-b^2}}{\sqrt{3b^2-1}}$$

# Notation

$M^n$ a $C^\infty$ n-dim manifold, TM tangent bundle

$\pi : TM \to M$ projection

$(x^1, ..., x^n)$ local coordinates on $U \subset M$.

$\frac{\partial}{\partial x^i}$ and $dx^i$ coordinate basis for $T_xM$ and $T_x^*M$

$(x, y)$ be a point of TM, $x \in M, y \in T_xM$

$(x^i, y^i)$ local coordinates on $\pi^{-1}(U) \subset TM$, where

$$y = y^i \frac{\partial}{\partial x^i}$$

# Finsler Space

$F : TM \longrightarrow [0, \infty)$ is called a Finsler metric on $M$ if $F$ has the following properties:

[i] (Regularity) $F \in C^{\infty}$ in $TM \setminus \{0\}$;

[ii] (Positive Homogeneity)

$$F(x, ty) = tF(x, y), \ \forall t > 0, (x, y) \in TM;$$

[iii] (Strong Convexity)

$$g = (g_{ij}(x, y)) = \left(\frac{1}{2}[F^2(x, y)]_{y_i y_j}\right)$$

is positive definite at each point of $TM \setminus \{0\}$.

The pair $(M, F)$ is called a Finsler space.

## Special Finsler Spaces:

- Minkowski space: $V^n$ an n-dimensional real vector space with a Minkowski norm F where $F(x, y)$ depends only on y.
- A Randers metric on M

$$F(x, y) = \alpha(x, y) + \beta(x, y),$$

$$\alpha(x, y) = \sqrt{a_{ij}(x) y^i y^j}, \quad \beta(x, y) = b_k(x) y^k,$$

$a_{ij}$ components of a Riemannian metric
$a^{ij}$ inverse matrix
$b_k$ components of the 1-form $\beta$, whose norm

$$b = \sqrt{a^{ij} b_i b_j}$$

satisfies $0 \le b < 1$.

## Finsler volume form

$(M^n, F)$ a Finsler space, then F induces a smooth volume form

$$d\mu_F = \sigma(x) dx_1 \wedge \dots \wedge dx_n$$

where

$$\sigma(x) = \frac{\mathrm{vol}\,(B^n)}{\mathrm{vol}\,\{y \in R^n;\ F(x, \Sigma y^i \frac{\partial}{\partial x_i}) \leq 1\}}$$

$B^n$ = unitary ball in $R^n$

vol = Euclidean volume

$(\widetilde{M}^m, \tilde{F})$ a Finsler space

$\varphi : M^n \longrightarrow (\widetilde{M}^m, \widetilde{F})$ an immersion.

There is an induced Finsler metric on M,

$$F(x,y) = (\varphi^* \widetilde{F})(x,y) = \widetilde{F}(\varphi(x), \varphi_*(y)), \quad \forall\ (x,y) \in TM$$

$\varphi : M^n \longrightarrow (\widetilde{V}^{n+1}, \widetilde{F}_b)$, special Randers space.

$x = (x^\gamma),\ \ \gamma = 1, \cdots, n,$

$\varphi(x) = \left(\varphi^i(x^\varepsilon)\right) \in \widetilde{V},\ i = 1, \cdots, n+1, \quad z^i_\gamma = \dfrac{\partial \varphi^i}{\partial x^\gamma}.$

The volume form in the induced metric is

$$d\mu_F = \left(1 - b^2 A^{\tau\gamma} z^{n+1}_\tau z^{n+1}_\gamma\right)^{\frac{n+1}{2}} \sqrt{\det A}\, dx^1 \cdots dx^n,$$

where

$$A = (A_{\tau\gamma}) = \left(\sum_{i=1}^{n+1} z^i_\tau z^i_\gamma\right) \quad \text{and} \quad (A^{\tau\gamma}) = (A_{\tau\gamma})^{-1}$$

# Mean Curvature

Introduced by Z. Shen

$\varphi : M^n \longrightarrow (\widetilde{M}^m, \widetilde{F})$ an immersion in Finsler space.

$\varphi_t : M^n \longrightarrow (\widetilde{M}^m, \widetilde{F})$, $\quad t \in (-\varepsilon, \varepsilon)$ such that

$\varphi_0 = \varphi$ and $\varphi_t = \varphi$ outside a compact $\Omega \subset M$.

$F_t = \varphi_t^* \widetilde{F}$ induced metrics and $\tilde{X} = \frac{\partial \varphi_t}{\partial t}|_{t=0}$

Consider the function $V(t) = \int_\Omega d\mu_{F_t}$. Then

$$V'(0) = \int_M \mathcal{H}_\varphi(\tilde{X}) d\mu_F.$$

$\mathcal{H}_\varphi$ is the mean curvature of the immersion $\varphi$.

$\mathcal{H}_\varphi(v)$ depends linearly on v.

$\mathcal{H}_\varphi$ vanishes on $\varphi_*(TM)$.

The immersion is minimal when $\mathcal{H}_\varphi \equiv 0$.

# Mean Curvature

$$\mathcal{H}_\varphi(\tilde{\mathbf{X}})|_{\mathbf{x}} = \frac{\mathbf{d}}{\mathbf{dt}}\left(\log \sigma_{\mathbf{t}}(\mathbf{x})\right)_{|\mathbf{t}=0} - \mathbf{div}\,[\mathbf{P}_{\varphi_*}(\tilde{\mathbf{X}})]_{|\mathbf{x}}$$

where

$$\mathbf{P}_{\varphi_*}(\tilde{\mathbf{X}}) = \frac{\partial \log \sigma}{\partial \mathbf{z}^{\mathbf{i}}_\gamma}\,\tilde{\mathbf{X}}^{\mathbf{i}}\frac{\partial}{\partial \mathbf{x}_\gamma}$$

and $\mathbf{z}^{\mathbf{i}}_\gamma = \dfrac{\partial \varphi^{\mathbf{i}}}{\partial \mathbf{x}^\gamma}$. Then

$$\mathcal{H}_\varphi(\mathbf{v}) = \frac{1}{\sigma}\left\{\frac{\partial^2\sigma}{\partial \mathbf{z}^{\mathbf{i}}_\varepsilon \partial \mathbf{z}^{\mathbf{j}}_\eta}\frac{\partial^2\varphi^{\mathbf{j}}}{\partial \tilde{\mathbf{x}}^\varepsilon \partial \tilde{\mathbf{x}}^\eta} + \frac{\partial^2\sigma}{\partial \mathbf{x}^{\mathbf{j}} \partial \mathbf{z}^{\mathbf{i}}_\varepsilon}\frac{\partial \varphi^{\mathbf{j}}}{\partial \tilde{\mathbf{x}}^\varepsilon} - \frac{\partial \sigma}{\partial \mathbf{x}^{\mathbf{i}}}\right\}\mathbf{v}^{\mathbf{i}}$$

Whenever $(\mathbf{V}, \mathbf{F})$ is a Minkowsky space,

$$\mathcal{H}_\varphi(\mathbf{v}) = \frac{1}{\sigma}\left\{\frac{\partial^2\sigma}{\partial \mathbf{z}^{\mathbf{i}}_\varepsilon \partial \mathbf{z}^{\mathbf{j}}_\eta}\frac{\partial^2\varphi^{\mathbf{j}}}{\partial \tilde{\mathbf{x}}^\varepsilon \partial \tilde{\mathbf{x}}^\eta}\right\}\mathbf{v}^{\mathbf{i}}$$

# Completeness of a Finsler manifold

$(M, F)$ a Finsler manifold, ( $F$ posit. homog. )

$\sigma : [a, b] \longrightarrow M$ a piecewise differentiable curve.

The integral length of $\sigma$

$$L(\sigma) = \int_a^b F\left(\sigma, \frac{d\sigma}{dt}\right) dt.$$

$\Gamma_{(p_0, p_1)}$ is the set of all piecewise $C^\infty$ curves

$$\sigma : [a, b] \longrightarrow M, \text{ with } \sigma(a) = p_0, \ \sigma(b) = p_1.$$

Define a map $d : M \times M \longrightarrow [0, \infty)$ by

$$d(p_0, p_1) := \inf_{\sigma \in \Gamma_{(p_0, p_1)}} L(\sigma).$$

$(M, d)$ satisfies the two axioms of a metric space.

(i) $d(p_0, p_1) \geq 0$, equality holds $\Longleftrightarrow$ $p_0 = p_1$

(ii) $d(p_0, p_1) \leq d(p_0, p_2) + d(p_2, p_1)$.

If $F(x, ty) = |t|F(x, y), \forall\, t \in R$, (F absol. homog.)

(iii) $d(p_0, p_1) = d(p_1, p_0)$

Generically, the distance function d does not have the symmetry property.

A Finsler manifold $(M, F)$ is forward complete with respect to the distance function d if every forward Cauchy sequence converges in M.

$(M, F)$ is forward geodesically complete if every geodesic $\gamma(t)$, $a \leq t < b$, parametrized to have constant Finsler speed, can be extended to $(a, \infty)$.

Similarly, one defines a backward complete and backward geodesically complete Finsler space.

If F is absolutely homogeneous of degree one, then forward and backward geodesic completeness either both hold or both fail.
This is the case for Riemannian metrics.

## The differential equation of minimal hypersurface in a special Randers space

**Theorem:** $\varphi : M^n \longrightarrow (V^{n+1}, F_b)$ an immersion with local coordinates $(\varphi^j(x_\varepsilon))$ is minimal $\Longleftrightarrow$

$$\left\{\frac{(n^2-1)}{4}\frac{\partial B}{\partial z^i_\varepsilon}\frac{\partial B}{\partial z^j_\eta}C - \right.$$

$$\frac{n+1}{2}(1-B)\left(\frac{\partial^2 B}{\partial z^i_\varepsilon \partial z^j_\eta}C + \frac{\partial B}{\partial z^j_\eta}\frac{\partial C}{\partial z^i_\varepsilon} + \frac{\partial B}{\partial z^i_\varepsilon}\frac{\partial C}{\partial z^j_\eta}\right) +$$

$$\left.(1-B)^2\frac{\partial^2 C}{\partial z^i_\varepsilon \partial z^j_\eta}\right\}\frac{\partial^2 \varphi^j}{\partial x^\varepsilon \partial x^\eta}v^i = 0,$$

$\forall v = v^i e_i \in V^{n+1}$, $e_i$ canonical basis of $V^{n+1}$,

$$C = \sqrt{\det A}, \quad B = b^2 A^{\varepsilon\eta} z^{n+1}_\varepsilon z^{n+1}_\eta, \quad z^i_a = \frac{\partial \varphi^i}{\partial x^a},$$

$$A = (A_{\tau\gamma}) = \left(\sum_{i=1}^{n+1} z^i_\tau z^i_\gamma\right) \qquad (A^{\tau\gamma}) = (A_{\tau\gamma})^{-1}.$$

**Remark:** If $\varphi$ is a minimal then $\tilde{\varphi} = \lambda\varphi$ is minimal

**Corollary:** $\varphi : M^2 \longrightarrow (V^3, F_b)$ an immersion, with local coordinates $(\varphi^j(x^\varepsilon))$. $\varphi$ is minimal $\Longleftrightarrow$

$$\left\{\frac{12E^2-(2E+C^2)^2}{C(C^2-E)}\frac{\partial C}{\partial z^i_\varepsilon}\frac{\partial C}{\partial z^j_\eta}-\frac{3C}{2}\frac{\partial^2 E}{\partial z^j_\eta \partial z^i_\varepsilon}-\right.$$

$$\frac{3}{2}\left(\frac{2E-C^2}{C^2-E}\right)\left(\frac{\partial C}{\partial z^i_\varepsilon}\frac{\partial E}{\partial z^j_\eta}+\frac{\partial C}{\partial z^j_\eta}\frac{\partial E}{\partial z^i_\varepsilon}\right)+$$

$$\left.\frac{3C}{4(C^2-E)}\frac{\partial E}{\partial z^i_\varepsilon}\frac{\partial E}{\partial z^j_\eta}+\frac{(2E+C^2)}{2C}\frac{\partial^2 C^2}{\partial z^j_\eta \partial z^i_\varepsilon}\right\}\frac{\partial^2\varphi^j}{\partial x^\varepsilon \partial x^\eta}v^i=0,$$

$\forall\, v = v^i e_i \in V^3$, where $C=\sqrt{\det A}$, $\quad z^i_\gamma = \dfrac{\partial \varphi^i}{\partial x^\gamma}$

$$A=(A_{\tau\gamma})=\left(\sum_{i=1}^{n+1} z^i_\tau z^i_\gamma\right) \qquad (A^{\tau\gamma})=(A_{\tau\gamma})^{-1}.$$

$$E=b^2\sum_{k=1}^{3}(-1)^{\gamma+\tau} z^k_{\tilde\gamma} z^k_{\tilde\tau} z^3_\gamma z^3_\tau, \qquad \tilde\tau=\begin{cases}1 & \text{if } \tau=2,\\ 2 & \text{if } \tau=1.\end{cases}$$

## Minimal surface generated by rotating a plane curve around a fixed axis.

**Theorem.** An immersion into $(V^3, F_b)$ given by

$$\varphi(t,\theta) = (f_b(t)\cos\theta,\ f_b(t)\sin\theta,\ t), \qquad f_b > 0.$$

is minimal $\Longleftrightarrow$ $f_b$ satisfies

$$-f_b f_b'' \left[\left(1 - b^2 + (f_b')^2\right)\left(1 + 2b^2 + (1-3b^2)(f_b')^2\right) + 3b^4 (f_b')^2\right] + \left(1 + (f_b')^2\right)\left(1 - b^2 + (f_b')^2\right)\left[1 - b^2 + \left(1 - 3b^2\right)(f_b')^2\right] = 0.$$

- Whenever $b = 0$, $V^3$ is the Euclidean space, we get the classical differential equation for minimal surfaces of rotation in $R^3$.

The mean curvature vanishes on tangent vectors of the immersion $\varphi$, hence we only need to consider v such that $\{v, \varphi_t, \varphi_\theta\}$ is lin. ind .
We consider

$$v = (-\cos\theta,\ -\sin\theta,\ f_b'(t)).$$

With the notation $x^1 = t$, $x^2 = \theta$ and $z_\varepsilon^i = \dfrac{\partial \varphi^i}{\partial x^\varepsilon}$,
we have

$$z_\varepsilon^3 = \delta_{\varepsilon 1} \quad \varphi^3_{x^\varepsilon x^\eta} = 0,\ \forall \varepsilon, \eta.$$

$$A = \begin{pmatrix} 1 + [f_b'(t)]^2 & 0 \\ 0 & [f_b(t)]^2 \end{pmatrix},$$

$$C^2 = [f_b(t)]^2[1 + [f_b'(t)]^2],$$

$$E = b^2[f_b(t)]^2.$$

## Existence and uniqueness of solutions for the equation

The diff. eq. for $f_b(t)$ can be rewritten as

$$\begin{cases} \dot{x}_1 & = x_2 \\ x_1 \dot{x}_2 Q_b(x_2) & = P_b(x_2), \end{cases}$$

where $x_1(t) = f_b(t)$ and $x_2(t) = \dot{x}_1(t)$,

$$P_b(x_2) = (1 + x_2^2)(1 - b^2 + x_2^2)[1 - b^2 + (1 - 3b^2)x_2^2],$$

$$Q_b(x_2) = (1 - b^2 + x_2^2)[1 + 2b^2 + (1 - 3b^2)x_2^2] + 3b^4 x_2^2.$$

## Remarks

- If $0 \le b \le \frac{\sqrt{3}}{3}$, then $P_b(x_2) > 0$ and $Q_b(x_2) > 0$.

- If $\frac{\sqrt{3}}{3} < b < 1$, then :

$P_b(\pm N_1(b)) = 0$, where $N_1(b) = \sqrt{\dfrac{1-b^2}{3b^2-1}}$.

$Q_b(\pm N_2(b)) = 0$, where

$$N_2(b) = \sqrt{\frac{1-b^2+3b^4+b^2\sqrt{12-12b^2+9b^4}}{3b^2-1}}$$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $P_b(x_2)$ | − | − | − | 0 | + | + | 0 | − | − | − |
| $Q_b(x_2)$ | − | 0 | + | + | + | + | + | + | 0 | − |
| $x_2$ | | $-N_2(b)$ | | $-N_1(b)$ | | 0 | $N_1(b)$ | | $N_2(b)$ | |

There are no solutions for initial conditions

$x_1(t_0) = a \neq 0, \quad x_2(t_0) = \pm N_2(b)$.

**Remarks:**

If $f_b(t)$ is a solution then $\frac{1}{c}f_b(a+ct)$, $c \neq 0$, is also a solution

We only need to consider two cases:

Case 1: $0 \leq b \leq \frac{\sqrt{3}}{3}$, with initial conditions

$$f_b(0) = a > 0 \quad \text{and} \quad f_b'(0) = d \in \mathbf{R};$$

Case 2: $\frac{\sqrt{3}}{3} < b < 1$, with initial conditions

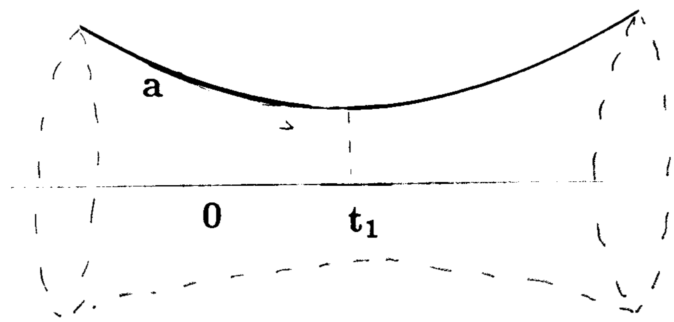
$$f_b(0) = a > 0 \quad \text{and} \quad f_b'(0) = d \neq \pm N_2(b).$$

**Lemma.** Let $0 \le b \le \frac{\sqrt{3}}{3}$ and $f_b(t)$ be the solution defined on the maximal interval J, satisfying the initial conditions $f_b(0) = a > 0, f_b'(0) = d \in R$, then

(i) $f_b(t) f_b''(t) > 0, \forall t \in J$;

(ii) there exists $t_1 \in J$ such that $f_b'(t_1) = 0$;

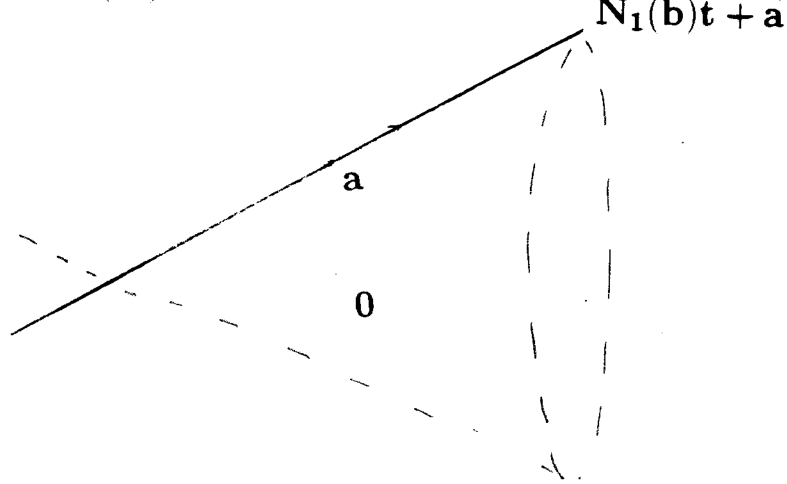
(iii) $f_b$ is symmetric with respect to the straight line $t = t_1$.




**Lemma.** Let $\frac{\sqrt{3}}{3} < b < 1$, and $f_b(t)$ be the solution with initial conditions $f_b(0) = a > 0, f_b'(0) = \pm N_1(b)$

Then

$$f_b(t) = \pm N_1(b)t + a$$



**Lemma.** Let $\frac{\sqrt{3}}{3} < b < 1$ and $f_b(t)$ be the solution defined on the maximal interval J, with initial conditions $f_b(0) = a > 0$, $f_b'(0) = d$, where $|d| < N_1(b)$. Then

(i) $|f_b'(t)| < N_1(b)$, $\forall t \in J$;

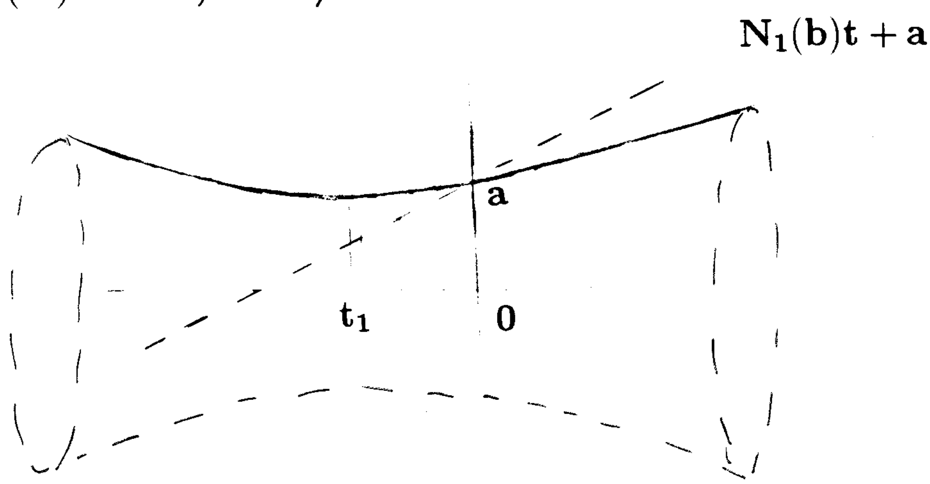
(ii) $f_b(t) f_b''(t) > 0$, $\forall t \in J$;

(iii) $J = (-\infty, \infty)$;

(iv) there exists $t_1 \in J$ such that $f_b'(t_1) = 0$;

(v) $f_b(t)$ is symmetric w. resp. to the line $t = t_1$;

(vi) the curve $f_b(t)$ does not intersect the line $\operatorname{sign}(d)\, N_1(b) t + a$, $\forall t \neq 0$.



**Lemma.** Consider $\frac{\sqrt{3}}{3} < b < 1$ and $f_b(t)$ the solution defined on the maximal interval J, with initial conditions $f_b(0) = a > 0,\ f_b'(0) = d,$ with $N_1(b) < |d| < N_2(b)$. Then

(i) $N_1(b) < |f_b'(t)| < N_2(b), \forall t \in J;$

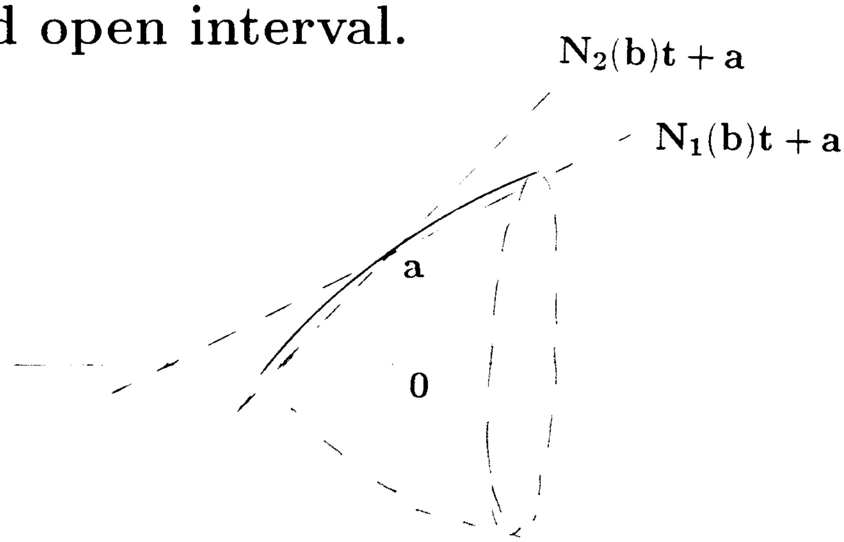
(ii) $f_b(t) f_b''(t) < 0, \forall t \in J;$

(iii) the curve $f_b(t)$, $t \neq 0$ is between the lines

$N_1(b)t + a$ and $N_2(b)t + a$ when $d > 0$

$-N_2(b)t + a$ and $-N_1(b)t + a$ when $d < 0$.

(iv) J is a bounded open interval.



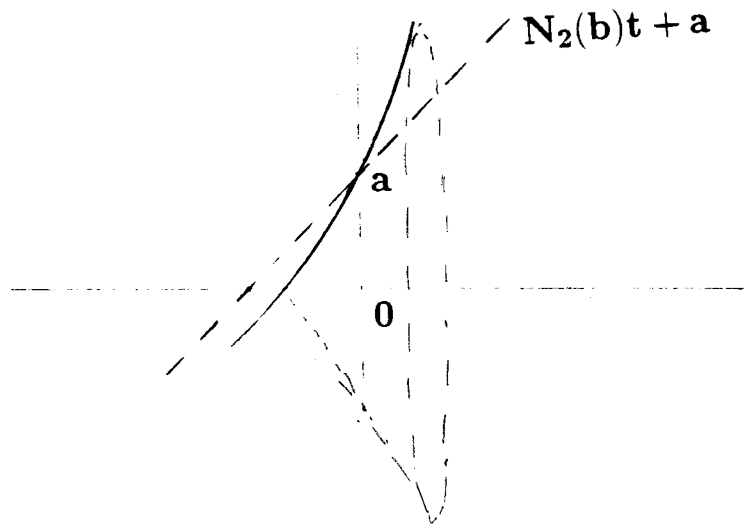
**Lemma.** Consider $\frac{\sqrt{3}}{3} < b < 1$ and $f_b(t)$ the solution defined on the maximal interval J, with initial conditions $f_b(0) = a > 0$, $f_b'(0) = d$, with $|d| > N_2(b)$. Then

(i) $|f_b'(t)| > N_2(b)$, $\forall t \in J$ ;

(ii) $f_b(t) f_b''(t) > 0$, $\forall t \in J$ ;

(iii) the curve $f_b(t)$ for $t \neq 0$ does not intersect the line $\text{sign}(d)\ N_2(b)t + a$.

(iv) J is bounded from below (resp. above) when $d > N_2(b)$ (resp. $d < -N_2(b)$).



**Proposition.** Consider the minimal surface of rotation $M^2$ in $(V^3, F_b)$ generated by the curve $(0, f_b(t), t)$, where $f_b$ is the solution with initial conditions: $f_b(0) = a > 0$ and $f_b'(0) = d$.

- If $0 \le b < \frac{\sqrt{3}}{3} \implies$ M is complete. (forward)
- If $\frac{\sqrt{3}}{3} < b < 1$ and

  $|d| < N_1(b) \implies$ M is complete. (forward)

  $|d| \ge N_1(b),\ |d| \neq N_2(b) \implies$ M is not complete.

## Remarks:

1. When $d = \pm N_1(b)$, then the surface is a cone generated by the straight line $f_b(t) = \pm N_1(b)t + a$.
2. The minimal cones converge to a cylinder when $b \to 1$, since $\lim_{b \to 1} N_1(b) = 0$.
3. For the complete surfaces we only need to consider initial conditions: $f_b(0) = a > 0$ and $f_b'(0) = 0$.