STOCHASTIC ANALYSIS AND FINSLER MANIFOLDS

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I. Diffusion on Finsler manifolds

1. Stochastic development – lifting Brownian motion to a Finsler manifold

2. Application in mathematical biology: Random perturbations of Volterra-Hamilton systems

3. Finslerian diffusion and curvature

4. Laplacian generated by a geodesic random walk on a Finsler manifold

5. Harmonic forms

II. Large deviations results involving Finsler metrics

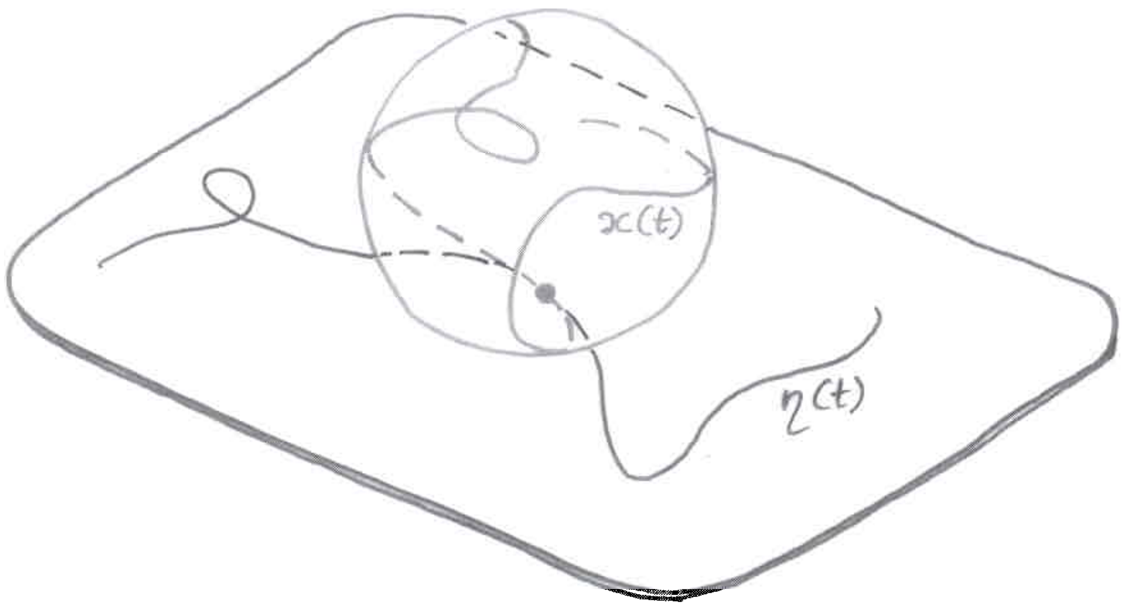
1. Asymptotics of stochastic differential equations with random coefficients

2. Application in mathematical finance

Stochastic Development

("rolling" a manifold along Brownian motion)

1. Riemannian Manifold M, $\dim M = d$



(a) deterministic case

$\eta(t)$ smooth curve in $\mathbb{R}^d$

$x(t)$ smooth curve on M

$z(t)$ horizontal lift of $x(t)$ to OM

$\dot{z} = \ell(\dot{x})$

$\ell(A)(x,z) = A^i(x)\left(\partial_i - \Gamma^k_{ij}(x)\, z^j_n \partial^n_k\right)$

A vect. field on M, $\ell(A)$ vect. field on OM

$$\boxed{\begin{cases} \dot{x} = z_i \dot{\eta}^i \\ \dot{z} = \ell(\dot{x}) \end{cases}}$$

development ("rolling") along η

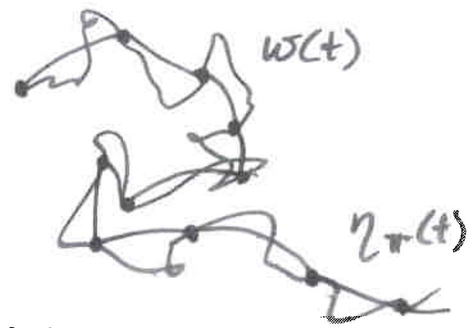
x independent of z

(b) stochastic case

$w(t)$ Brownian motion in $\mathbb{R}^d$

$\eta_\pi(t)$ piecewise smooth approximation of $w(t)$

$\pi: \ 0 = t_0 < t_1 < \ldots < t_n = T$



$$\begin{cases} \dot{x}_\pi = z_{\pi i} \dot{\eta}^i_\pi \\ \dot{z}_\pi = \ell(\dot{x}_\pi) \end{cases}$$ development along η_π

$(x_\pi, z_\pi) \to (x, z)$ in probability as mesh $\pi \to 0$

$$\boxed{\begin{cases} dx = z_i \circ dw^i \\ dz = \ell(\circ dx) \end{cases}}$$ development along BM w

(Stratonovich stochastic differential equations)

<u>Result:</u> $x(t)$ (without $z(t)$!) is Markovian with generator $D = \frac{1}{2}\Delta$

$$= \frac{1}{2} g^{ij}(\partial_i \partial_j - \Gamma^k_{ij}\partial_k)$$

<u>Def:</u> $x(t)$ called Brownian motion on M

Two extensions of stochastic development to Finsler manifolds

(a) h-stochastic development

$w(t)$ Brownian motion in $\mathbb{R}^d$

$x(t)$ diffusion on M

$y(t)$ horiz. lift of $x(t)$ to TM

$z(t)$ horiz. lift of $x(t), y(t)$ to OM

$$\boxed{\begin{cases} dx = z_i \circ dw^i \\ dy = h(\circ dx) \\ dz = \ell(\circ dy) \end{cases}}$$

(b) hv-stochastic development

$w(t), v(t)$ independent Brownian motions on $\mathbb{R}^d$

$x(t), y(t)$ diffusion on TM

$z(t)$ horiz. lift of $x(t), y(t)$ to OM

$$\boxed{\begin{cases} dx = z_i \circ dw^i \\ dy = h(\circ dx) + z_i \circ dv^i \\ dz = \ell(\circ dy) \end{cases}}$$

Both approaches introduced by
Antonelli, Zastawniak 1993, 1994

Extension to Lagrange manifolds and vector bundles
Antonelli, Hrimiuc 1996

(c) diffusion of vectors, tensors, exterior forms on M

For example, for vectors

$$\boxed{dy = h(\circ dx)}$$

where

$$h(A)(x,y) = A^i(x)\left(\partial_i - \Gamma_{ij}^k(x) y^j \dot{\partial}_k\right)$$

horizontal lift of a vector field A to TM

2. Finsler manifold M, $\dim M = d$

$$\begin{tikzcd} TM \arrow[dr] & \\ & M \\ TM \arrow[uu, "A"] \arrow[ur] & \end{tikzcd}$$

$$\begin{aligned} h(A)(x,y) &= A^i(x,y)\left(\partial_i - F_{ij}^k(x,y) y^j \dot{\partial}_k\right) \\ &= A^i(x,y)\left(\partial_i - N_i^k(x,y)\, \dot{\partial}_k\right) \\ &= A^i(x,y)\, \delta_i \end{aligned}$$

$$\begin{tikzcd} TTM \arrow[d, bend left] \\ TM \arrow[u, "h(A)"] \end{tikzcd}$$

horizontal lift of a Finsler vector field A to a standard vector field $h(A)$ over TM

$$\begin{tikzcd} TTM \arrow[d, bend left] \\ TM \arrow[u, "B"] \end{tikzcd}$$

$B = A^i \delta^i + B^i \dot{\partial}_i$ vector field over TM

$$\begin{aligned} \ell(B)(x,y,z) = {} & A^i(x,y)\left(\delta_i - F_{ij}^k(x,y)\, z_n^j\, \partial_k^n\right) \\ & + B^i(x,y)\left(\dot{\partial}_i - C_{ij}^k(x,y)\, z_n^j\, \partial_k^n\right) \end{aligned}$$

$$\begin{tikzcd} TOM \arrow[d, bend left] \\ OM \arrow[u, "\ell(B)"] \arrow[d] \\ TM \end{tikzcd}$$

horizontal lift of a standard vector field B over TM to a vector field $\ell(B)$ over OM, the Finsler orthonormal frame bundle over TM

Here $(N_i^k, F_{ij}^k, C_{ij}^k)$ Cartan connection

$$x \in M, \quad y \in TM_x, \quad z \in OM_{x,y}$$

Application in mathematical biology: Random perturbations of Volterra-Hamilton systems

N^i size of population of species i

x^i production variable (biomass)

$$\begin{cases} \dfrac{dx^i}{dt} = N^i & \text{production equation} \\[2ex] \dfrac{dN^i}{dt} = -F^i_{jk}(x,N)\dfrac{dN^j}{dt}\dfrac{dN^k}{dt} + \lambda N^i & \text{ecological equation} \end{cases}$$

F^i_{jk} ecological interactions between species

Substitute $N^i = \dfrac{dx^i}{dt}$ and $s = \dfrac{1}{\lambda}e^{\lambda t}$ in the second equation to get

$$\boxed{\frac{d^2x^i}{ds^2} + F^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0}$$

Def: <u>Volterra-Hamilton system of Finsler type</u> if this is the geodesic equation in a Finsler metric $L(x,y)$

Examples of Volterra–Hamilton systems of Finsler type

$$\begin{cases} \dfrac{dx^i}{ds} = y^i \\ \dfrac{dy^i}{ds} + F^i_{jk}(x,y)\, y^j y^k = 0 \end{cases}$$

Example: Growth of a colony
e.g. two-coral colony

$$L(x,y) = \sqrt{(y^1)^2 + (y^2)^2}\; e^{Q \arctan(y^1/y^2) + (Q^2+1)(\alpha_1 x^1 + \alpha_2 x^2)}$$

$$\begin{cases} \dfrac{dy^1}{ds} + 2(\alpha_2 + Q\alpha_1)\, y^1 y^2 + (\alpha_1 - Q\alpha_2)\left[(y^1)^2 - (y^2)^2\right] = 0 \\ \dfrac{dy^2}{ds} + 2(\alpha_1 - Q\alpha_2)\, y^1 y^2 + (\alpha_2 + Q\alpha_1)\left[(y^2)^2 - (y^1)^2\right] = 0 \end{cases}$$

Antonelli Berwald metric

Example: Host-parasite system with social interactions
e.g. rabbit-flea model for myxomatosis

$$L(x,y) = \left[(y^P)^\mu + (y^H)^\mu\right]^{1/\mu} e^{\gamma_P x^P - \gamma_H x^H}$$

$$\begin{cases} \dfrac{dy^P}{ds} = -\gamma_P (y^P)^2 + \gamma_H \dfrac{\mu}{\mu-1}\, y^P y^H + \dfrac{\gamma_P}{\mu-1}\left(\dfrac{y^H}{y^P}\right)^{\mu-2} (y^H)^2 \\ \dfrac{dy^H}{ds} = \gamma_H (y^H)^2 + \gamma_P \dfrac{\mu}{\mu-1}\, y^P y^H - \dfrac{\gamma_H}{\mu-1}\left(\dfrac{y^P}{y^H}\right)^{\mu-2} (y^P)^2 \end{cases}$$

Example: Bacterial colony exchanging chemicals

$$L(x,y) = y^2 \left(\frac{y^2}{y^1}\right)^{1/\lambda} e^{-\alpha_1 x^1 + (\lambda+1)\alpha_2 x^2 + \nu x^1 x^2}$$

$\alpha_1, \alpha_2, \lambda > 0,\ \nu \neq 0$

$$\begin{cases} \dfrac{dy^1}{ds} + \lambda(\alpha_1 - \nu x^2)(y^1)^2 = 0 \\ \dfrac{dy^2}{ds} + \lambda\left(\alpha_2 + \dfrac{\nu}{\lambda+1} x^1\right)(y^2)^2 = 0 \end{cases}$$

Adding noise to a Volterra-~~Hamilton~~ system

$$\begin{cases} \dfrac{dx}{ds} = y \\ \dfrac{dy}{ds} = h\left(\dfrac{dx}{ds}\right) \end{cases}$$

deterministic system

Rule for noise addition:

The infinitesimal Finslerian distance by which the state (x, y) of the system will be displaced should be proportional to the magnitude of the perturbation dw, dv

So:

1. Unroll deterministic system to a straight line (constant velocity)
2. Perturb position and velocity by Brownian motions w and v
3. Roll back

$$\boxed{\begin{cases} dx = y\, ds + z_i \circ dw^i \\ dy = h(\circ dx) + z_i \circ dv^i \\ dz = l(\circ dy) \end{cases}}$$

perturbed system

(x,y) Markov diffusion with generator D

$\rho(s,x,y)$ probability density of $(x(s), y(s))$

$$\boxed{\begin{aligned} D^*\rho &= \frac{\partial \rho}{\partial s} \\ \rho(0,x,y) &= \varphi(x,y) \\ \rho(s,x,y)\Big|_{y^1y^2=0} &= 0 \end{aligned}}$$

in $y>0$

initial condition at $s=0$

boundary condition

Solution by Feynman-Kac formula

$$D^* = D + A + V$$

A vector field on TM

V scalar field on TM

$$\boxed{\rho(s,x,y) = \mathbb{E}_{s,x,y}\left[\varphi(\tilde{x}(0),\tilde{y}(0))\, e^{\int_0^s V(\tilde{x}(u),\tilde{y}(u))\,du}\right]}$$

where $(\tilde{x},\tilde{y})$ diffusion with drift A

$\mathbb{E}_{s,x,y}$ conditional expectation given that $\tilde{x}(s)=x$, $\tilde{y}(s)=y$, $y^1y^2>0$.

Finslerian Diffusion and Curvature

Berwald space $\delta_l C_{ijk} = 0 \Leftrightarrow D^i_{jkl} = 0 \Leftrightarrow G^i_{jk}$ indep. of y

$G^i_{jk} = \dot\partial_j \dot\partial_k (\gamma^i_{jk} y^j y^k)$ Berwald connection coefficients

$D^i_{jkl} = \dot\partial_j G^i_{kl}$ Douglas tensor

$C_{ijk} = \frac{1}{2} \dot\partial_k g_{ij}$ Cartan torsion tensor

In Berwald spaces there exist normal coordinates (NC)

Def: For h-diffusion $x(t), y(t)$ define

$$\mathcal{U}_\varepsilon(x(t), y(t)) = g_{ij}(x(t), y(t)) \mathbb{E}_t \left[x^i(t+\varepsilon) x^j(t+\varepsilon) \right] \quad \text{in NC}$$

called Quadratic Dispersion

Theorem: In Berwald space

$$\mathcal{U}_\varepsilon = \frac{\dim M}{2} \varepsilon + \frac{R}{12} \varepsilon^2 + o(\varepsilon^2)$$

where R is the scalar curvature

(Antonelli, Zastawniak 1997)

In Riemannian case $\frac{R}{12}$ is the Onsager–Machlup term
(Onsager, Machlup 1953 ; Takahashi, Watanabe 1981)

Cartan's Lemma for Berwald spaces

$$-g^{jk} \delta_i F^i_{jk} = g^{hk} g^{im} \delta_h \delta_k g_{ij} = \frac{2}{3} R \quad \text{in NC}$$

Theorem: for $h\nu$-diffusion $\frac{R}{12}$ replaced by $\frac{R+S}{12}$

Laplacian generated by a geodesic random walk

Riemannian case : Pinsky ~ 1980

M compact Finsler manifold, positive definite metric

$\xi_t(x,y)$ geodesic with $\xi_0(x,y) = x \in M$
$\dot{\xi}_0(x,y) = y \in IM_x$

$$(T_t^0 f)(x,y) = f(\xi_t(x,y), \dot{\xi}_t(x,y)) \quad , \quad f \in C^\infty(IM)$$

semigroup with generator

$$Z = y^i \partial_i - F^i_{jk}(x,y) y^j y^k \dot{\partial}_i = y^i \delta_i \qquad \text{geodesic flow field on } IM$$

IM_x compact Riemannian manifold with metric induced by the diagonal lift metric
$g_{ij}\, dx^i \otimes dx^j + g_{ij}\, \delta y^i \otimes \delta y^j$ on TM

ω_x volume measure on IM_x normalised to 1.

$$(\pi f)(x) = \int_{IM_x} f(x,y)\, d\omega_x(y) \quad , \quad \pi : C^\infty(IM) \to C^\infty(M) \text{ projection}$$

$e_1, e_2, \ldots$ i.i.d. random variables with exponential distribution $P\{e_n > t\} = \exp(-t)$, $t \geq 0$

$t_n = e_1 + \cdots + e_n$ jump times of a Poisson process

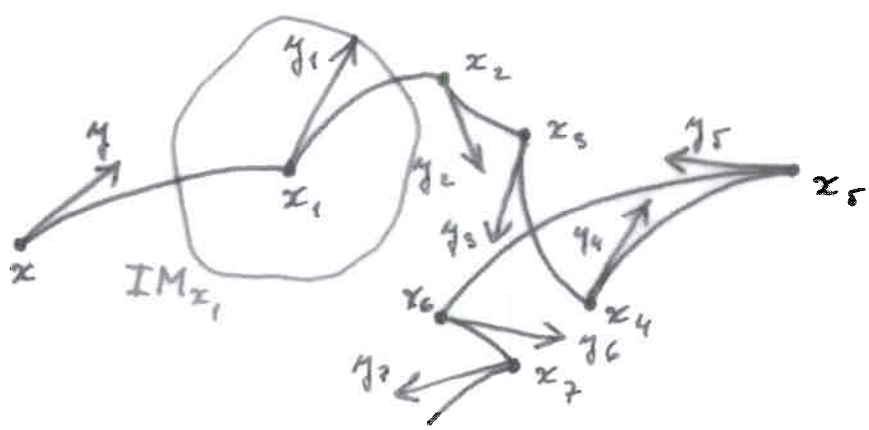
Construction of geodesic random walk

$x_0 = x \ , \quad y_0 = y$

$x_1 = \xi_{t_1}(x_0, y_0)\ , \quad y_1$ random var. uniformly distr. on IM_{x_1} independent of t_1

$x_2 = \xi_{t_2}(x_1, y_1)\ , \quad y_2$ random var. uniformly distr. on IM_{x_2} independent of y_1, t_1, t_2

$\vdots$



$$X_t(x,y) = \xi_{t-t_n}(x_n, y_n)\ , \quad Y_t(x,y) = \dot{X}_t(x,y)$$
$$\text{for} \quad t \in [t_n, t_{n+1})\ , \quad n = 0,1,2,\ldots$$

$$(T_t f)(x,y) = f(X_t(x,y), Y_t(x,y))\ , \quad f \in C^\infty(IM)$$

semigroup with generator

$$Lf = \lim_{t \searrow 0} \frac{T_t f - f}{t} = Zf + \Pi f - f$$

Central Limit Theorem

$\varepsilon > 0$

$X_t^\varepsilon, Y_t^\varepsilon$ geodesic random walk scaled so that mean time and distance between jumps are ε^2 and ε

(speed $1/\varepsilon$, jump rate $1/\varepsilon^2$)

$(T_t^\varepsilon f)(x,y) = f(X_t^\varepsilon(x,y), Y_t^\varepsilon(x,y))$ semigroup

$L^\varepsilon = \frac{1}{\varepsilon} Z + \frac{1}{\varepsilon^2}(\Pi - 1)$ generator

Theorem: $X_t^\varepsilon \xrightarrow{\text{weakly}} W_t$ as $\varepsilon \searrow 0$

i.e. $\lim_{\varepsilon \searrow 0} \mathbb{E}(f(X_t^\varepsilon)) = \mathbb{E}(f(W_t)) \quad \forall f \in C^\infty(M)$

where W_t Markov diffusion with generator

$$(\Pi Z^2 f)(x) = \int_{IM_x} y^i y^j (\partial_i \partial_j - F_{ij}^k(x,y)\partial_k)\, d\omega_x(y)$$

Antonelli, Zastawniak 1998

Def: A-Z Laplacian $\Delta_{AZ} = \Pi Z^2$

$$(\Delta_{AZ} f)(x) = \int_{IM_x} y^i y^j (\partial_i \partial_j - F_{ij}^k(x,y)\partial_k)\, d\omega_x(y)$$

for $f \in C^\infty(M)$

W_t called Brownian motion on Finsler manifold M

Harmonic Forms

$$H^{ij}(x) = \int_{IM_x} y^i y^j \, \omega_x \qquad \left(= \frac{g^{ij}(x)}{\dim M} \text{ if } M \text{ Riemannian} \right)$$

ω_x normalised volume form on IM_x

scalar product of p-forms φ, ψ

$$(\varphi, \psi) = \int_{IM} \varphi_{i_1 \dots i_p} \psi_{j_1 \dots j_p} H^{i_1 j_1} \dots H^{i_p j_p} \sqrt{g} \, \omega_x \wedge dx^1 \wedge \dots \wedge dx^n$$

$$= \int_M \varphi_{i_1 \dots i_p} \psi_{j_1 \dots j_p} H^{i_1 j_1} \dots H^{i_p j_p} \underbrace{\left(\int_{IM_x} \sqrt{g} \, \omega_x \right)}_{\sqrt{G}} dx^1 \wedge \dots \wedge dx^n$$

$$(d\varphi, \psi) = (\varphi, d^* \psi)$$

Harmonic forms $\psi \in \mathcal{H}^p M \quad (d^* d + d d^*) \psi = 0$

Hodge decomposition $\quad \Lambda^p M = \mathcal{H}^p M \oplus d \Lambda^{p-1} M \oplus d^* \Lambda^{p+1} M$

Bao-Lackey:

$$(\varphi, \psi) = \int_{SM} \varphi_{i_1 \dots i_p} \psi_{j_1 \dots j_p} g^{i_1 j_1} \dots g^{i_p j_p} \sqrt{g} \, \omega_x \wedge dx^1 \wedge \dots \wedge dx^n$$

$$= \int_M \varphi_{i_1 \dots i_p} \psi_{j_1 \dots j_p} G^{i_1 \dots i_p j_1 \dots j_p} \sqrt{G} \, dx^1 \wedge \dots \wedge dx^n$$

$$G^{i_1 \dots i_p j_1 \dots j_p} = \frac{1}{\sqrt{G}} \int_{SM_x} g^{i_1 j_1} \dots g^{i_p j_p} \sqrt{g} \, \omega_x$$

Centore:

$$(\varphi, \psi) = \int_M \varphi_{i_1 \dots i_p} \psi_{j_1 \dots j_p} K^{i_1 j_1} \dots K^{i_p j_p} \omega(x) \, dx^1 \wedge \dots \wedge dx^n$$

$$K^{ij}(x) = \frac{\int_{L(x,y) \le 1} y^i y^j \, dy}{\int_{L(x,y) \le 1} dy}, \qquad \omega(x) = \frac{\int_{|y| \le 1} dy}{\int_{L(x,y) \le 1} dy}$$

dy Euclidean volume form in TM_x

Asymptotics of stochastic differential equations with random coefficients

$$dX = b_{\lambda(t)}(X)\,dt + \sigma_{\lambda(t)}(X)\,dW_t$$

W_t Brownian motion in $\mathbb{R}^m$

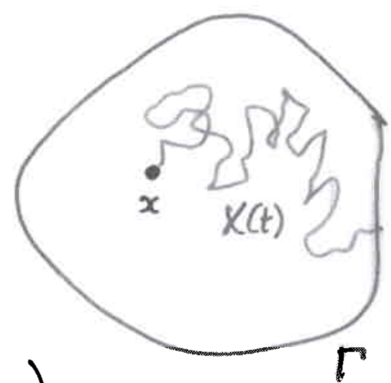
$\lambda(t)$ Markov process with values in state space $\Lambda = \{1, \dots, k\}$ assumed "connected"

$b_\lambda, \sigma_\lambda$ Lipschitz for each $\lambda \in \Lambda$ with values in $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$

$a_\lambda = \frac{1}{2}\sigma_\lambda \sigma_\lambda^*$ positive definite

$D \subset \mathbb{R}^n$ domain with boundary Γ

initial cond $\quad X(0) = x \in D$
$\qquad\qquad\quad\ \lambda(0) = \lambda \in \Lambda$



$u_\lambda(t,x) = \mathrm{Prob}(X \text{ hits } \Gamma \text{ before time } t)$

Theorem (large deviations result, Lasry + Lions 1995)

$$-2t \log u_\lambda(t,x) \to L^2(x) \quad \text{as } t \searrow 0$$

uniformly on compact sets in D

where $\quad L(x) = \inf\{L(x,y) : y \in \Gamma\}$

$$L(x,y) = \inf_{\lambda \in \Lambda} L_\lambda(x,y) \qquad \text{Finsler metric}$$

$L_\lambda(x,y)$ Riemannian metric with metric tensor a_λ^{-1}

Theorem (classical large deviations result, Varadhan 1967)

For $\quad dX = b(X)\,dt + \sigma(X)\,dW_t$

$-2t \log u(t,x) \to L(x)$ Riemannian distance to boundary, metric tensor a^{-1}, $a = \frac{1}{2}\sigma\sigma^*$

Application in mathematical finance

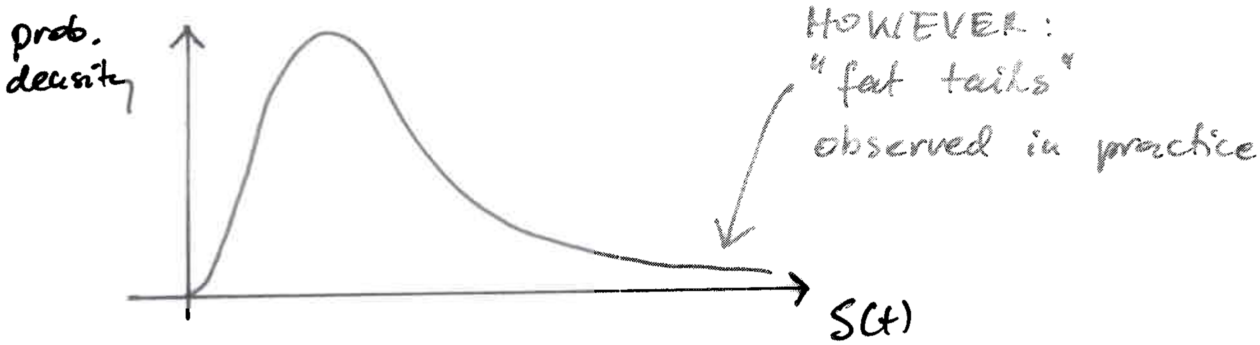
Standard Black-Scholes model of stock price S

$$dS = \mu S\,dt + \sigma S\,dW_t$$

μ rate of return, σ volatility, assumed constant

solution $$S(t) = S(0)\exp\left(\mu t - \tfrac{1}{2}\sigma^2 t + \sigma W_t\right)$$

log normal distribution of stock price
(i.e. $\log S(t)$ is normally distributed)



Suppose volatility non-constant, jumping between $\sigma_1 < \sigma_2$ according to Markov process

$$\begin{tikzcd}
\sigma_1 \arrow[loop left, "(1-\frac{c_{12}}{\varepsilon})dt"] \arrow[rr, bend left, "\frac{c_{12}}{\varepsilon}dt"] & & \sigma_2 \arrow[ll, bend left, "\frac{c_{21}}{\varepsilon}dt"] \arrow[loop right, "(1-\frac{c_{21}}{\varepsilon})dt"]
\end{tikzcd}$$

stationary probability $\dfrac{c_{21}}{c_{12}+c_{21}}$, $\dfrac{c_{12}}{c_{12}+c_{21}}$

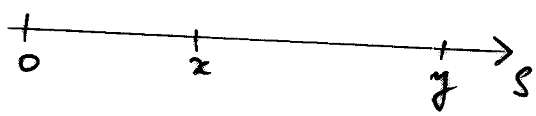
Consider two regimes

1) t fixed, $\varepsilon \downarrow 0$ — frequent jumps regime

"average" volatility observed

$$\bar{\sigma}^2 = \frac{c_{21}\sigma_1^2 + c_{21}\sigma_2^2}{c_{12} + c_{21}}$$

2) ε fixed, $t \downarrow 0$ — large deviations / short time regime



$D = (0, y)$

$\Gamma = \{y\}$

$$L(x) = L(x,y) = \inf_{\lambda = 1,2} L_\lambda(x,y)$$

$$L_\lambda(x,y) = \inf_{\substack{\gamma(0)=x \\ \gamma(1)=y}} \int_0^1 \sqrt{\frac{1}{\sigma_\lambda^2 \gamma(t)^2} \dot{\gamma}(t)^2}\, dt = \frac{1}{\sigma_\lambda} \left| \ln y/x \right|$$

$$\sigma_1 < \sigma_2 \Rightarrow L_1(x,y) > L_2(x,y)$$

$$u(t,x) \sim e^{-\frac{\ln^2 y/x}{2t\sigma_2}}$$

higher volatility σ_2 observed