A SCHUR LEMMA FOR EINSTEIN RANDERS METRICS

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June 7, 2002

Ricci curvature and Einstein metrics

The relevant quantities are

- the spray curvature, $K^i{}_j := y^h R_h{}^i{}_{jk} y^k$;
- the Ricci scalar Ric, given by $K^i{}_i = Ric\, F^2$;
- the Ricci tensor, $\mathrm{Ric}_{ij} := \left[\frac{1}{2} K^s{}_s\right]_{y^i y^j}$.

DEFINITION: F is **Einstein** if $Ric(x,y)$ is a function of x alone. Equivalently,

$$\mathrm{Ric}_{ij} = Ric(x) g_{ij} \,,$$

where $g_{ij} := \frac{1}{2}(F^2)_{y^i y^j}$.

GOAL: Show Ric is constant for Einstein metrics of Randers type, $n > 2$.

NOTATION: Randers metrics are denoted $F = \alpha + \beta$, where

$$\alpha = \sqrt{\tilde{a}_{ij}(x) y^i y^j} \,, \qquad \beta = \tilde{b}_i(x) y^i \,.$$

Schur fails in 2-dimensions

Consider a surface of revolution $M \subset \mathbb{R}^3$, parametrised as

$$(\vartheta, \varphi) \;\mapsto\; (\, f(\varphi)\cos(\vartheta),\; f(\varphi)\sin(\vartheta),\; g(\varphi)\,)\,.$$

Shen perturb by $W := \epsilon\, \partial_\vartheta$, $\epsilon |f| < 1$. The resulting Randers metric $F = \alpha + \beta$,

$$\alpha = \frac{\sqrt{u^2\, f^2 + v^2\,(1-\epsilon^2 f^2)\,(\dot f^2 + \dot g^2)}}{1 - \epsilon^2 f^2}\,,$$

$$\beta = \frac{-\epsilon\, u\, f^2}{1-\epsilon^2 f^2}\,, \qquad y = (u,v) \in T_{(\vartheta,\varphi)} M\,.$$

is Einstein, with Ricci scalar

$$Ric = \frac{\dot g\,(\dot f\, \ddot g - \ddot f\, \dot g)}{f\,(\dot f^2 + \dot g^2)^2}\,,$$

a non-constant function of φ.

The Riemannian Schur Lemma

Assume $\widetilde{\mathrm{Ric}}_{ij} = Ric(x)\tilde{a}_{ij}$.

All Riemannian surfaces satisfy this condition. Suppose $n > 2$.

Trace on (i, j) to obtain

$$\tilde{S} := \mathrm{Ric}^i{}_i = Ric(x)n \quad \Rightarrow \quad \widetilde{\mathrm{Ric}}_{ij} = \frac{\tilde{S}}{n}\tilde{a}_{ij}\,.$$

The second Bianchi identity:

$$0 = \tilde{R}_h{}^i{}_{jk|l} + \tilde{R}_h{}^i{}_{lj|k} + \tilde{R}_h{}^i{}_{kl|j}$$

Trace on (i, j) and (h, l):

$$\begin{aligned} 0 &= 2\widetilde{\mathrm{Ric}}^i{}_{k|i} - \tilde{S}_{|k} \\ &= \frac{2}{n}\tilde{S}_{|k} - \tilde{S}_{|k} \\ \Rightarrow \quad 0 &= (n-2)\tilde{S}_{|k} \end{aligned}$$

Whence $Ric_{|k} = \frac{1}{n}\tilde{S}_{|k} = 0$ when $n > 2$.

A Finslerian Obstacle

The second Bianchi identity for a Finsler metric:

$$\begin{aligned} R_h{}^i{}_{jk|l} + R_h{}^i{}_{lj|k} + R_h{}^i{}_{kl|j} \\ = P_h{}^i{}_{js} R^s{}_{kl} + P_h{}^i{}_{ks} R^s{}_{lj} + P_h{}^i{}_{ls} R^s{}_{jk} \, , \end{aligned}$$

where

$$R^s{}_{jk} := \frac{1}{F} y^i R_i{}^s{}_{jk} \, .$$

The non-vanishing right hand side leads one to expect the Schur Lemma to fail for general Finsler metrics.

Special tensors

Covariant differentiation by the Levi-Civita connection of $\tilde{a}_{ij}$ is denoted by a vertical slash:

$$\begin{aligned} \tilde{b}_{i|j} &:= \tilde{b}_{i,x^j} - \tilde{b}_h \tilde{\gamma}^h{}_{ij} \\ \mathrm{lie}_{ij} &:= \tilde{b}_{i|j} + \tilde{b}_{j|i}\ , \\ \mathrm{curl}_{ij} &:= \tilde{b}_{i|j} - \tilde{b}_{j|i}\ , \\ \theta_i &:= \tilde{b}^h \mathrm{curl}_{hi}\,. \end{aligned}$$

The characterization of Einstein Randers metrics $F = \alpha + \beta$

The Basic Equation

$$\mathrm{lie}_{ik} = \sigma(\tilde{a}_{ik} - \tilde{b}_i \tilde{b}_k) - \tilde{b}_i\, \theta_k - \tilde{b}_k\, \theta_i$$

The Curvature Equation

$$\begin{aligned} \widetilde{\mathrm{Ric}}_{ij} \quad &= \quad \big(\tilde{a}_{ij} + \tilde{b}_i \tilde{b}_j\big) Ric(x) \\ &\quad - \tfrac{1}{4} \tilde{a}_{ij}\, \mathrm{curl}^{hk} \mathrm{curl}_{hk} \\ &\quad - \tfrac{1}{2} \mathrm{curl}^h{}_i \mathrm{curl}_{hj} \\ &\quad - (n-1) \Big\{ \tfrac{1}{16} \sigma^2 \big(3\tilde{a}_{ij} - \tilde{b}_i \tilde{b}_j\big) \; + \; \tfrac{1}{4} \theta_i \theta_j \\ &\qquad\qquad + \tfrac{1}{4} \big(\theta_{i|j} + \theta_{j|i}\big) \Big\} \end{aligned}$$

The E(23) Equation

$$\begin{aligned} \mathrm{curl}^h{}_{i|h} \quad &= \quad 2\, Ric(x)\, \tilde{b}_i \; + \\ &\quad (n-1) \left\{ \tfrac{1}{8} \sigma^2 \tilde{b}_i + \tfrac{1}{2} \sigma \theta_i + \tfrac{1}{2} \mathrm{curl}^h{}_i \theta_h \right\} \end{aligned}$$

Matsumoto's Identity for Einstein Randers metrics: Preliminary form

Matsumoto's identity for constant curvature Randers metrics, $\sigma(K + \frac{1}{16}\sigma^2) = 0$, may be generalised to Einstein metrics:

$$n \{1 - \|\tilde{b}\|^2\} \, \sigma \, \left(K + \tfrac{1}{16}\sigma^2\right) \, + \, 2K_{|\tilde{b}} = 0 \,,$$

where $K = \frac{1}{n-1} Ric$.

Proof: The Ricci Identity for curl_{ij},

$$\mathrm{curl}_{ij|k|h} - \mathrm{curl}_{ij|h|k} = \mathrm{curl}_{sj} \, \tilde{R}_i{}^s{}_{kh} \, + \, \mathrm{curl}_{is} \, \tilde{R}_j{}^s{}_{kh} \,.$$

Trace (i,k) and (h,j),

$$\mathrm{curl}^{ij}{}_{|i|j} = \mathrm{curl}^{ij} \widetilde{\mathrm{Ric}}_{ij} = 0 \,.$$

With the Basic and E(23) equations, $\mathrm{curl}^{ij}{}_{|i|j} = 0$ yields the identity.

A Schur Lemma for Einstein Randers Spaces

ASSUME : $F = \alpha + \beta$, $\|\tilde{b}\| < 1$, is Einstein with Ricci scalar $Ric = Ric(x)$. In particular, the Basic, E(23) and Curvature Equations, and the preliminary form of Matsumoto's Identity hold.

CLAIM : The Ricci scalar satisfies $Ric_{|k} = 0$ for $n > 2$.

STRATEGY: Apply the Riemannian second Bianchi identity to the Einstein Curvature Equation.

2nd Bianchi and Einstein Curvature Eqs

$$
\begin{aligned}
0 &= \widetilde{\mathrm{Ric}}^{i}{}_{i|k} - 2\widetilde{\mathrm{Ric}}^{i}{}_{k|i} \\
&= (2\,\theta_k - \mathrm{lie}^{i}{}_{i}\,\tilde{b}_k)\,Ric + \,(n + \|\tilde{b}\|^2 - 2)\,Ric_{|k} \\
&\quad -2\,Ric_{|\tilde{b}}\,\tilde{b}_k - \mathrm{curl}^{ij}{}_{|i}\mathrm{curl}_{jk} \\
&\quad -\tfrac{1}{2}\,n\,\mathrm{curl}^{ij}\mathrm{curl}_{ij|k} + \mathrm{curl}^{ij}\mathrm{curl}_{ik|j} \\
&\quad +(n-1)\,\Big\{ \tfrac{1}{16}\,\sigma^2\,(2\,\theta_k - \mathrm{lie}^{i}{}_{i}\,\tilde{b}_k) + \\
&\qquad \tfrac{1}{2}\theta^{i}{}_{|i}\theta_k + \tfrac{1}{2}\theta^{i}(\theta_{k|i} - \theta_{i|k}) \,+ \\
&\qquad \tfrac{1}{2}(\theta^{i}{}_{|k|i} + \theta_{k}{}^{|i}{}_{|i} - \theta^{i}{}_{|i|k}) \Big\}
\end{aligned}
$$

We need to understand:

(T1) $-\frac{1}{2}\,n\,\mathrm{curl}^{ij}\mathrm{curl}_{ij|k} + \mathrm{curl}^{ij}\mathrm{curl}_{ik|j}$;

(T2) $\theta^{i}{}_{|i}$,

(T3) $\theta^{i}(\theta_{k|i} - \theta_{i|k})$, and

(T4) $\frac{1}{2}(\theta^{i}{}_{|k|i} + \theta_{k}{}^{|i}{}_{|i} - \theta^{i}{}_{|i|k})$.

A Formula for $\text{curl}_{ij|k}$

At this point it is helpful to derive the following identity:

$$\boxed{\text{curl}_{ij|k} = -2\tilde{b}^s \tilde{R}_{ksij} + \text{lie}_{ik|j} - \text{lie}_{kj|i}\,.}$$

Label this expression as (DC).

The displayed equality follows from the Ricci identity for $\tilde{b}$ and the definition of lie_{ij}:

$$\begin{aligned}
\tilde{b}_{i|j|k} - \tilde{b}_{i|k|j} &= \tilde{b}^s \tilde{R}_{isjk} \\
\tilde{b}_{i|k|j} + \tilde{b}_{k|i|j} &= \text{lie}_{ik|j} \\
-\tilde{b}_{k|i|j} + \tilde{b}_{k|j|i} &= -\tilde{b}^s \tilde{R}_{ksij} \\
-\tilde{b}_{k|j|i} - \tilde{b}_{j|k|i} &= -\text{lie}_{kj|i} \\
\tilde{b}_{j|k|i} - \tilde{b}_{j|i|k} &= \tilde{b}^s \tilde{R}_{jski}\,.
\end{aligned}$$

Summing the five equalities above and applying the first Bianchi identity produces highlighted equation above.

(T1): The first application of (DC)

With the previous formula for $\mathrm{curl}_{ij|k}$ and the skew-symmetry of curl^{ij} we may show

$$\mathrm{curl}^{ij}\mathrm{curl}_{ik|j} = \tfrac{1}{2}\mathrm{curl}^{ij}\mathrm{curl}_{ij|k}\,.$$

Hence we may rewrite (T1) as

$$\begin{aligned} -\tfrac{1}{2}\,n\,\mathrm{curl}^{ij}\mathrm{curl}_{ij|k} &+ \mathrm{curl}^{ij}\mathrm{curl}_{ik|j} \\ &= \quad -\tfrac{1}{2}\,(n-1)\,\mathrm{curl}^{ij}\mathrm{curl}_{ij|k}\,. \end{aligned}$$

(T2): A formula for ${\theta^i}_{|i}$

Notice that

$$\tilde{b}_{h|i}\mathrm{curl}^{hi} = \tfrac{1}{2}(\mathrm{lie}_{hi} + \mathrm{curl}_{hi})\mathrm{curl}^{hi} = \tfrac{1}{2}\mathrm{curl}_{hi}\mathrm{curl}^{hi}$$

A calculation with the E(23) Equation reveals

$$\begin{aligned}
{\theta^i}_{|i} &= (\tilde{b}_h \mathrm{curl}^{hi})_{|i} \\
&= \tilde{b}_{h|i}\mathrm{curl}^{hi} + \tilde{b}_h {\mathrm{curl}^{hi}}_{|i} \\
&= \tfrac{1}{2}\mathrm{curl}_{hi}\,\mathrm{curl}^{hi} + \tfrac{1}{2}(n-1)\,\theta_i\,\theta^i \\
&\quad - \{2Ric + \tfrac{1}{8}(n-1)\,\sigma^2\}\,\|\tilde{b}\|^2 .
\end{aligned}$$

A second application of (DC)

With the Basic Equation and (DC) compute

$$
\begin{aligned}
\theta_{k|i} - \theta_{i|k} &= (\tilde{b}^j \mathrm{curl}_{jk})_{|i} - (\tilde{b}^j \mathrm{curl}_{ji})_{|k} \\
&= (\tilde{b}^j{}_{|i} \mathrm{curl}_{jk} + \tilde{b}^j \mathrm{curl}_{jk|i}) \\
&\quad - (\tilde{b}^j{}_{|k} \mathrm{curl}_{ji} + \tilde{b}^j \mathrm{curl}_{ji|k}) \\
&\overset{(\mathrm{DC})}{=} \tfrac{1}{2}(\mathrm{lie}^j{}_i + \mathrm{curl}^j{}_i)\mathrm{curl}_{jk} \\
&\quad - \tfrac{1}{2}(\mathrm{lie}^j{}_k + \mathrm{curl}^j{}_k)\mathrm{curl}_{ji} \\
&\quad \tilde{b}^j \Big\{ - 2b^s(\tilde{R}_{isjk} - \tilde{R}_{ksji}) \\
&\qquad + (\mathrm{lie}_{ji|k} - \mathrm{lie}_{ik|j}) \\
&\qquad - (\mathrm{lie}_{jk|i} - \mathrm{lie}_{ki|j}) \Big\} \\
&\overset{(\mathrm{BE})}{=} -\sigma \mathrm{curl}_{ki}
\end{aligned}
$$

This formula for the skew-symmetric part of $\theta_{k|i}$, will be used to compute (T3) and (T4).

(T3): The first application of $\theta_{k|i} - \theta_{i|k} = -\sigma \text{CURL}_{ki}$

The term (T3) is given by :

$$\theta^i(\theta_{k|i} - \theta_{i|k}) = \sigma\, \theta^i\, \text{curl}_{ik}$$

(T4): The second application of $\theta_{k|i} - \theta_{i|k} = -\sigma\mathrm{CURL}_{ki}$

The Ricci Identity for θ implies

$$\theta^i{}_{|k|i} = \theta^i{}_{|i|k} + \theta^i \widetilde{\mathrm{Ric}}_{ik} \, .$$

It follows that

$$\begin{aligned} \theta_k{}^{|i}{}_{|i} &= (\theta^i{}_{|k} + \sigma \mathrm{curl}^i{}_k)_{|i} \\ &= \theta^i{}_{|i|k} + \theta^i \widetilde{\mathrm{Ric}}_{ik} + \sigma \mathrm{curl}^i{}_{k|i} \, . \end{aligned}$$

Now we see the last term (T4) may be rewritten as

$$\begin{aligned} \tfrac{1}{2}\left(\theta^i{}_{|k|i} + \theta_k{}^{|i}{}_{|i} - \theta^i{}_{|i|k}\right) &= \\ &\tfrac{1}{2}\,\theta^i{}_{|i|k} + \theta^i \widetilde{\mathrm{Ric}}_{ik} + \tfrac{1}{2}\sigma \mathrm{curl}^i{}_{k|i} \, . \end{aligned}$$

Finale

- Substitute the derived formulas for the terms T1, T2, T3 and T4 into the second Bianchi expression.
- After applying the Basic, E(23) and Curvature Equations, and Matsumoto's Identity; and utilizing the expression for $\theta_{k|i} - \theta_{i|k}$ we have

$$\begin{aligned} 0 &= \tilde{S}_{|k} - 2\widetilde{\mathrm{Ric}}^{i}{}_{k|i} \\ &= (n-2)(1 - \|\tilde{b}\|^2)\, Ric_{|k} \end{aligned}$$

Hence $Ric_{|k} = 0$, and $Ric(x)$ is constant.

Q.E.D.

Comments

- Matsumoto's identity for (y-global) Einstein Randers metrics is updated to

$$\sigma(K + \tfrac{1}{16}\sigma^2) = 0\,,$$

  when $n > 2$. The identity now agrees with the constant curvature version.

- The Einstein characterisation is essential. The Finsler second Bianchi identity is not amenable to a Schur type argument. The characterisation allows us to by-pass the Randers Ricci tensor Ric_{ij} and work with the Riemannian $\widetilde{\mathrm{Ric}}_{ij}$ and second Bianchi identity.

- Open Question: Does the Schur Lemma hold for arbitrary Finsler metrics?