Tight closure.

Top "ten" list of why STU

11. Shortens hairy proofs
10. Gives analysts lice?
9. Proves Briançon-Skoda theorem
8. Proves ring of invariants are Cohen-Macaulay
7. Proves local homological conjectures
6. Eliminates unsightly cohomology (Jacobian ideal)
5. Makes vanishing theorems
4. Makes contracted extension "bulge" manageable
3. Is Res
2. Is 2
1. Is finite

⇒ splitting theorems

3. Controls your Cohen ideals
   \((x_1, \ldots, x_k): x_{k+1} \subseteq (x_1, \ldots, x_k)\)
   
   where the \(x\)'s are a system of parameters

2. Compensates failure of regularity
1. If \(R\) is regular, it does nothing:
   \(R\) regular, \(I^* = I\)

Tight closure is in regular rings is a tool for proving elements are in ideals.
Let \( k[x_{ij}] \) is a \( m \times m \) \( i \leq j \leq m \)

\[ R = \frac{k[x_{ij}]}{I(x_{ij})} \]

Every ideal of \( R \) is tightly closed, but for \( \ell \neq 1 \) \( R \) is not regular. \( R \) will be a localization of finitely generated algebra over an excellent ring local ring containing a field.

\( x \in I^\# (x \in \overline{I}) \iff \) this is true mod \( P \) for all minimal primes \( P \) of \( R \).

So \( R \) is a noetherian domain

1. \( x \in \overline{I} \iff \) \( x \) maps \( R \xrightarrow{\phi} V \) where \( V \) valuation ring \( f(x) \in IV \)

If \( R \) is noetherian it is enough to look at \( D \), \( R/S \) being DVR when \( R \) is a domain it is enough to check \( R \xrightarrow{\phi} V \).

2. \( x \in \overline{I} \iff \) it satisfies an equation: \( x^m + x_{1}x_{1}^m + \ldots + x_{m} = 0 \)

where the \( x_{j} \in I^j \) \( (\Rightarrow x \in \sqrt{I}) \)

3. \( x \in \overline{I} \iff \exists c \neq 0 \text{ in } R \) (\( R \) is a domain) \( \Rightarrow x_{m} \in I^m \) \( \forall m \gg 0 \)

(\( c \) or \( m \) for infinitely many \( m \))

1., 2., 3. are definitions of integral closure.
Briançon–Skoda Theorem

If \( R \) is regular equal char 0 and \( I \) has \( m \) generators
then \( \overline{I^m} \subseteq I \)

If \( x \in I^m \Rightarrow x \in \overline{I} \) (just take \( x_m = -x^m \) \( \frac{m}{m+1} \) and \( x^m x_m = 0 \)
is the right equation)
then, if
\[
(u, v) \subseteq I \subseteq R \quad (u, v)_j = (u^j, v^j) \in I \subseteq I \quad I^{(i)} = I^m, (i+j)=m
\]
then \( \overline{(u, v)} = \overline{(u^m, v^m, u^{m-1}, v, \ldots, v^m)} \)

If \( R = C[z, \ldots, z_n] \), \( \Omega \) is a "mixed" mbd of the origin

- Briançon–Skoda for \( R \):
  If \( f, g_1, \ldots, g_k \) are holomorphic functions in \( \Omega \)
  \( X = \) common zeros of \( g_1, \ldots, g_k \)
  \[
  \sum_{i} \frac{1}{\text{local power}} \int_{\Omega \setminus X} \left( \frac{|g_1|^2 + \ldots + |g_k|^2}{\text{local other power}} \right) \text{stuff} \, dV < \infty
  \]
  \( 
  \Rightarrow \quad f = \sum_i g_i h_i \quad h_i \text{ holomorphic in } \Omega
  \)

- Hilbert's Nullstellensatz:
  If \( f \) vanishes where all the \( g_i's \) vanish \( \Rightarrow \)
  \( f \in \text{Rad}(g_1, \ldots, g_k) \)
R is noetherian, char p
Let q = p^e

**Def.** R is a noetherian domain. x ∈ R, I ⊆ R ideal of R then x ∈ I* if ∃ c ∈ R such that cx^q ∈ I[c^q] for all q > 0 (all q is equivalent).

where I[c^q] = (c^q | i ∈ I) = (f_i^q | i ∈ I) where I = (f_1,...,f_k)

(Σ x_j i_j)^q = Σ x_j q i_j^q

I[c^q] ⊆ I^q

**Easy**
1. I ⊆ I*
2. I ⊆ J =⇒ I* ⊆ J*
3. (I*)* = I*

If R ⊆ S take x ∈ IS \ R \ I x is "almost" in I

module finite

Let I = (f_1,...,f_k)

 cx^q = Σ x_j f_j^q
 cf_1 = Σ x_j f_j ∈ IS, Σ = R[c^q, x_1^q, ..., x_k^q]

s is module finite over R

**Theorem (Brionoug - Skoda)**

I m generated : I^m ⊆ I* (= Brionoug skoda for regular ring containing a field of char p).
pf: $R$ domain

$u \in I^m$, where $I = (f_1, \ldots, f_m)$

$\Rightarrow c \cdot u^m \in (I^m)^m \land m, c \neq 0$

$\Rightarrow cu^q \in (I^m)^q \land q$

$\Rightarrow (f_1, \ldots, f_m)^{mq} \subseteq (f_1^q, \ldots, f_m^q) = I^{[q]}$

Also:

**THM** if $R$ is regular $\Rightarrow I = I^* \land I$ ideal of $R$.

**pf:** suppose $R$ is a domain $c \neq 0$, $cm^q \in I^{[q]}$ all $q$.

Let $F: R \to R$, $F^e: R \to R$

$r \to x^p \quad x \to x^p = x^q$.

if $R$ is regular then $F$ is flat.

(the proof reduces to the local, complete case

$k[1, x_1, \ldots, x_k] \cong k[1, x_1^p, \ldots, x_k^p] \cong k[1, x_1, \ldots, x_k]$)

In this case it is easy to show that the Frobenius map is flat).

**Lemma** $S$ is flat over $R$, $ISR \in R$

$IS: S \in (I: R)S$

**pf: exercise.** Hint: $\frac{R \cdot x}{I} \to \frac{R}{I}$ and tensor with $S$.

back to the Theorem:

$I^{[q]} = IS \quad R \xrightarrow{F^e} R$

$I^{[q]} : x^q = (I : x)^{[q]} \quad \text{if } R \text{ is regular.}$

$c \in \cap I^{[q]} : x^q = \cap (I : x)^{[q]} \subseteq \cap (I : x)^q = 0$ unless $I : x = R$

i.e. $x \in I$