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Tight closure. 2.

How do you know \( x \in I^* \)?
This is an hard problem because you need to check infinitely many equations \( cx^pe \in I^{(p^e)} \) for infinitely many \( e \).

Example:

\[
\frac{k[x,y,z]}{(x^3+y^3+z^3)} \quad \text{char } k \neq 3
\]

we want to show \( z^2 \in (x,y)^* \) but \( z \notin (x,y)(z^2 \neq 0 \text{ m} \nabla \)

\[
\frac{k[x,y,z]}{(x, y, x^3+y^3+z^3)} \cong \frac{k[z]}{z^3}
\]

For the \( z^2 \in (x,y)^* \) I need to check the following:

CLAIM

\[
x(z^2)^q \in (x,y) \quad \text{[eq]} \quad (x^q, y^q)
\]

\( q \equiv a \text{ mod } 3 \), \( q=1 \text{ or } q=2 \)

\( 2q = 6h + 2a \) since we can write \( q=3h+a \)

\[
xz^2q = xz^{6h+2a} = z^a (x^3+y^3)^h z^b \text{ write } q= \frac{3}{h} + b
\]

\[
= \pm z \left( \sum h \cdot x^{3i} y^{3j} \right) z^b
\]

either \( 3i+1 \geq q \)

or \( 3j \geq q \)

if \( 3i+1 \leq q-1 \)

and \( 3j \leq q-1 \)

\( \Rightarrow 3h+1 < 2q-2 \Rightarrow 2q > 3h+3 \times \)

\( \Rightarrow x(z^2)^q \in (x^q, y^q) \text{ for } q \gg 0 \)
Testing whether \( u \in \mathbb{I}^* \) in
\[
\frac{k[x,y,z]}{(x^2+y^3+z^3)}
\]
is a very hard problem:

if \( R \) is a finitely generated \( \mathbb{Q} \)-algebra and \( I \subseteq R \) how do you test tight closure?

Find a finitely generated \( \mathbb{Z} \)-algebra \( A \subseteq R \) s.t. \( x \in A \), \( J = I \cap A \) generates \( I \), such that \( \mathbb{Q} \otimes_{\mathbb{Z}} A = R \)

(example \( \mathbb{Q} \otimes \mathbb{Z}[x,y,z] = \mathbb{Q}[x,y,z] \)).

**Def.** Define \( x \in \mathbb{I}^* \) if for almost all primes \( p \) the image of \( x \) is \( m ( J_{/pA}^*) \) (in the Cherlin p sense)

(Use locally excellent rings that contains \( \mathbb{Q} \)).

**Example:**
\[
\begin{align*}
\mathbb{Q} \otimes \mathbb{Z}[x,y,z] \quad &\longrightarrow\quad \mathbb{Z}_p \otimes \mathbb{Z}[x,y,z] \\
\frac{k[x,y,z]}{(x^3+y^3+z^3)} \quad &\longrightarrow\quad \frac{k[x,y,z]}{(x^3+y^3+z^3)}
\end{align*}
\]

we just checked in the previous example that

\( 3, z^2 \in (x,y)^* \), in the example almost all \( q \)

where \( \mathbb{Q}_p \).

the definition is independent by the choice of \( A \).

going back to charp:

if \( x \in \mathbb{I}^* \), \( I \subseteq R \) ideal
\( R \rightarrow S \) any ring homomorphism
\( f(x) \in (IS)^* \) over \( S \).

This is called Persistence, the proof is not easy.
Def of tight closure for locally excellent noetherian rings containing $\mathbb{Q}$.

$x \in R, I \subseteq R$

$x \in I^*$ if for all maps $R \to B$, $B$ is a complete local domain.

If a finitely generated $\mathbb{Q}$-algebra $T$ and a map $T \to B$ and $x_0 \in T$, $I_0 \subseteq T$

s.t. $x_0 \in I_0^*$ in $T$ and $x_0 \mapsto f(x)$ in $B$, and $I_0B = IB$.

there are a lot of open questions about the char $0$ def.

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**THM** (Ein-Lazarsfeld-Smith comparison thm)

$R$ a field. $P$ is an height $h$ prime of $R$, $R$ is regular.

Then: $p(hm) \subseteq p^m \forall m$.

where $p^{(m)} = p^m R_p \cap R$

Thus it is also true if $R_p$ has finite projective dim.

(hint: $pa R_p \to \infty \Rightarrow p^{(m)}$ is unmixed.)

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**THM** (Hochster - Roberts)

If $R$ is a noetherian $k$-algebra, regular. $G$ is a linearly reductive algebraic group over $k$,

acting over $R \Rightarrow R^G$ is Cohen Macaulay

\[ \text{rang of m\-variants} \]

There is an $R^G$-linear map from $R$ to $R^G$, i.e. $R^G \to R$ splits as a map of $R^G$-modules.
THM: (Hochster-Huneke)

If \( S \) is regular, \( R \rightarrow S \) splits as a map of \( R \)-modules

\( \Rightarrow \) \( R \) is Cohen-Macaulay

(the proof reduces to the case where \( R \) is local and complete).

* Cohen capturing:

**THM** If \( R \) is a local domain "nice" (for example excellent)

and let \( x_1, \ldots, x_{k+1} \) part of a system of parameters:

\( (x_1, \ldots, x_k) : x_{k+1} \subseteq (x_1, \ldots, x_k)^* \)

**pf:** let \( R \) be a complete local domain. Let

\( k \) be a coefficient field of \( R \) then:

\( k[x_1, \ldots, x_k, \ldots, x_m] \otimes \mathbb{A} \)

\( A \subseteq R \)

module finite

\( u \cdot \mathbb{A}x_{k+1} \in (x_1, \ldots, x_k) \) claim: \( u \in (x_1, \ldots, x_k)^* \)

choose \( A^h \subseteq R \) free and pick \( c \in A^h \) s.t.

\( c \cdot R \subseteq A^h \)

\( (u^q x_{k+1}) \in (x_1^q, \ldots, x_k^q) R \)

\( cu^q x_{k+1} \in (x_1^q, \ldots, x_k^q) A^h \)

\( \Rightarrow cu^q \in (x_1^q, \ldots, x_k^q) A^h \) but \( A^h \subseteq R \)

\( \Rightarrow cu^q \in (x_1^q, \ldots, x_k^q) R \) all \( q \). \( \Box \)
**THM**  
R is a direct summand of $S$, $S$ regular. $\Rightarrow$  
R is Cohen Macaulay

**pf:**  
R complete local, $S = R \otimes W$ as $R$-modules.  
If $u \in I^* m R$, $u \in (IS)^* = IS$. $u \in R \cap IS$  
$R \otimes 0 \cap (I \otimes IW) = I$  
m$_{R} I = I^*$ for all I  

$$(x_1, \ldots, x_k), x_{k+1} \subseteq (x_1, \ldots, x_k)^* = (x_1, \ldots, x_k)$$