Maschke's Theorem

If \( \text{char} \, k \neq 1|G| \) then every short exact sequence of \( kG \)-modules splits.

\[ \text{so } \text{Ext}^{>0}_{kG} (-, -) = 0 \]

In particular \( H^{>0}(G, k) = 0 \).
Computing Cohomology of a Finite Group

Fact $|G|$ annihilates $H^*(G, \mathbb{Z})$, so only need to compute $p$-torsion for each $p | |G|$.

Transfer Map tells us that if $S$ is a Sylow $p$-subgroup of $G$ then $\text{res}_{G,S}: H^*(G, \mathbb{Z}) \rightarrow H^*(S, \mathbb{Z})$ is injective.

Cartan-Eilenberg Stable Element Method

The image of $\text{res}_{G,S}$ consists of elements such that if $g \in G$ and $H^g \subseteq S$ then
\[ \text{conjug} \circ \text{res}_{S,H^g} = \text{res}_{S,H^g} \]
"Stable elements w.r.t. fusion in $G"
If $G$ is a $p$-group,

**By Hand** 1) Do cyclic groups using explicit resolutions

2) Use induction on order. Choose a central cyclic subgroup $1 \neq \mathbb{Z} \leq G$ and use either

the LHS Spectral sequence

$$H^*(G/\mathbb{Z}, H^*(\mathbb{Z}, k)) \Rightarrow H^*(G, k)$$

or the Eilenberg-Moore spectral sequence

$$\text{Tor}^{**}_k(k, H^*(G/\mathbb{Z}, k)) \Rightarrow H^*(G, k)$$

The latter was used by Rusin to compute mod $2$ cohomology of groups of order $32$; the former was used by Quillen to compute cohomology of extraspecial $2$-groups.
machine computation (Carlson)

Compute an explicit free resolution of $k$ as $kG$-module. Don't build the modules, just write the maps as matrices with entries in $kG$.

Calculating when to stop resolving: use some commutative algebra! (see lecture 3)

Carlson computed the mod 2 cohomology of the groups of order 64 this way.
Example Normal abelian $A$

Sylow $p$-subgroups in char $p$

$1 \rightarrow A \rightarrow G \rightarrow G/A \rightarrow 1$

$LHS$ Spectral Sequence

collapses to give

$H^*(G,k) = H^*(A,k)^{G/A}$

The inclusion $H^*(G,k) \hookrightarrow H^*(A,k)$

splits (as an $H^*(G,k)$-module)

so $H^*(G,k)$ is Cohen-Macaulay
Associated Primes

& Steenrod Operations

Group cohomology with \( \mathbb{F}_p \) coefs admits an action of the Steenrod operations \((i \geq 0)\):

\[
\begin{align*}
Sq^i : H^n(BG; \mathbb{F}_2) & \rightarrow H^{n+i}(BG; \mathbb{F}_2) \quad (p=2) \\
P^i : H^n(BG; \mathbb{F}_p) & \rightarrow H^{n+2i(p-1)}(BG; \mathbb{F}_p) \quad (p \text{ odd})
\end{align*}
\]

These satisfy "Adem relations".

Steenrod algebra is generated by Steenrod operations, modulo Adem rels.

As an algebra over the Steenrod algebra \( H^*(BG; \mathbb{F}_p) \) contains a copy of the Dickson invariants as a h.s.o.p.

\[
\mathbb{F}_p[x_1, \ldots, x_r]^{GL_r(\mathbb{F}_p)} = \mathbb{F}_p[c_{r,r-1, \ldots, c_{r,0}}]
\]

dilated via \( 1x; 1 = 2^p \cdot \\
1c_{r,1} : 1 = 2(p^{a+r} - p^{a+i}). \)
The Landweber-Stong conjecture proved by Bourguiba & Zarati says that for any unstable module over the Steenrod algebra, which is simultaneously a module over the Dickson invariants with a suitable compatibility condition (for example $H^*(BG; \mathbb{F}_p)$) the depth is the largest $d$ for which $c_{r_1} c_{r_2} \cdots c_{r_{r_1}} \cdots c_{r_r}$ is a regular sequence. i.e., we can use a single test sequence!

Conjecture (Carlson) Let $G$ be a finite group and $k$ a field. Then $H^*(G, k)$ has an associated prime whose dimension is the depth...
Theorem (Wilkerson)

If \( H \) is an unstable algebra over the Steenrod algebra
(i.e., \( Sq^i(x) = 0 \) for \( i > 1 \times 1 \) \( (p=2) \)) \( \hat{P}^i(x) = 0 \) for \( i > 2(p-1) \times 1 \) \( (p \text{ odd}) \))

e.g. mod \( p \) cohomology of a space

Then the radical of the annihilator of any element
is Steenrod invariant.

In particular,

associated primes are

Steenrod invariant
Theorem (Serre)

Let $E$ be an elementary abelian $p$-group. Then the Steenrod invariant prime ideals in $H^*(E, \mathbb{F}_p)$ are in 1-1 corr. with the subgroups of $E$. If $E' \subseteq E$ then the corresponding Steenrod invariant prime ideal is

\[ \sqrt{\ker \text{res}_{E,E'}}. \]

Combining with Quillen's inseparable isogeny, we get:
Theorem: $G$ compact Lie

The Steenrod invariant prime ideals in $H^*(BG; \mathbb{F}_p)$ are the ideals of the form

$$\sqrt{\ker \text{res}_{G,E}}$$

where $E$ is an elementary abelian $p$-subgroup of $G$.

In particular, the associated primes are of this form, but for which elementary abelian $p$-subgroups $E$? - Difficult question.
IDEMPOTENT MODULES

AND VARIETIES

$G$ finite group, char $k = p$

$V_G = \text{max ideal spectrum of } H^*(G, k)$

e.g. If $G \cong (\mathbb{Z}/p)^r$ then $V_G \cong \mathbb{A}^r(k)$

Quillen's inseparable isogeny gives

$$\lim_{E \in \mathcal{A}_p(G)} V_E \rightarrow V_G$$

is bijective at level of points (usually not invertible in category of varieties)
If $M$ is a \textit{finitely generated} $kG$-module, the kernel of

$$H^*(G, k) = \text{Ext}^*_{kG}(k, k)^\otimes M \to \text{Ext}^*_{kG}(M, M)$$

is an ideal in $H^*(G, k)$ which defines a closed homogeneous subvariety $V_G(M) \subseteq V_G$.

Get same answer from the ideal $\bigcap \text{Annihilator of Ext}^*_{kG}(S, M)$ is simple.
Properties of $V_G(M)$

(i) $V_G(M) = \{0\} \iff M$ is projective

(ii) $V_G(M \oplus N) = V_G(M) \cup V_G(N)$

(iii) $V_G(M \otimes_k N) = V_G(M) \cap V_G(N)$

(iv) If $0 \neq \xi \in H^*(G, k)$ is represented by $\xi : \Omega^* k \to k$ with kernel $L_{\xi}$, then $V_G(L_{\xi})$ is the hypersurface determined by $\xi$ in $V_G$, denoted $V_G(\langle \xi \rangle)$.

For any closed bgs $V \subseteq V_G$ can choose $\xi_1, \ldots, \xi_t$ with

$V = V_G(\langle \xi_1 \rangle) \cup \ldots \cup V_G(\langle \xi_t \rangle) = V_G(L_{\xi_1}) \cap \ldots \cap V_G(L_{\xi_t}) = V_G(L_{\xi_1} \otimes_k \ldots \otimes_k L_{\xi_t})$. 

$\square$
INFINITE DIMENSIONAL MODULES

Why?
1. Various constructions naturally give infinite dimensional modules
2. Representing objects for functors are often infinite dim.

Analogy
In algebraic topology, even if you're only interested in finite CW complexes, cohomology is represented by Eilenberg-Mac Lane spaces, K-theory by BU, cobordism by MU, etc. These are all infinite complexes.
\[ \text{Mod}(kG) = \text{category of all } kG\text{-modules and homomorphisms} \]

\[ \text{StMod}(kG) : \text{same objects, morphisms} \]

\[ \text{Hom}_{kG}(M,N) = \text{Hom}_{kG}(M,N) / \text{PHom}_{kG}(M,N) \]

\[ \text{PHom}_{kG}(M,N) = \text{subspace of maps which factor through a projective } kG\text{-module} \]

\[ \text{StMod}(kG) \text{ is not an abelian category} - \text{can't tell whether a map is injective or surjective} \]

\[ \text{Instead it's triangulated} \]

\[ \text{Triangle } A \rightarrow B \rightarrow C \rightarrow \text{Si } A \text{ corresponds to a short exact sequence} \]

\[ 0 \rightarrow A \rightarrow B \oplus (\text{proj}) \rightarrow C \rightarrow 0 \]

\[ \text{mod}(kG), \text{ st mod}(kG) \text{ full subcategories of } \text{f.g. modules} \]
\( \text{Proj } H^*(G, k) = \text{set of closed} \)
\( \text{homogeneous irreducible subvarieties} \)
\( \text{of } V_G. \)
\( \text{If } \mathcal{W} \subseteq \text{Proj } H^*(G, k) \text{ is closed} \)
\( \text{under specialization (i.e.,} \)
\( \text{if } V \in \mathcal{W}, \ W \subseteq V \text{ then } W \in \mathcal{W} \) \)
\( \mathcal{M} = \text{full subcategory of} \)
\( \text{stmod } (kG) \text{ consisting of modules} \)
\( M \text{ s.t. } V_G(M) \text{ is a finite union} \)
\( \text{of elements of } \mathcal{W} \)
\( \text{then } M \text{ is a thick } \underline{\text{subcategory}} \)
\( \text{of } \text{stmod } (kG), \ \underline{\text{ideal closed}} \)
\( \text{To such a subcategory, Rickard} \)
\( \text{associates a triangle} \)
\[ E_\mathcal{W} \rightarrow k \rightarrow F_\mathcal{W} \rightarrow \Sigma^{-1} E_\mathcal{W} \]
\( \text{of idempotent modules} \)
Characterized by:

$E_Y$ is a filtered colimit of modules in $M$

For any $M$ in $M$, we have $\text{Hom}_{kG}(M, F_Y) = 0$

**Example** If $V = V_G<\xi> \in H^*(G, k)$ & $\mathcal{V}$ = set of subvarieties of $V$
write $E_\xi$, $F_\xi$ for $E_Y$, $F_Y$
Represent $\xi$ by a cocycle

$\hat{\xi} : \Sigma^n k \to k$ and dimension shift:

$$k \xrightarrow{\Sigma^n \hat{\xi}} \Sigma^2 k \xrightarrow{\Sigma^{2n} \hat{\xi}} \Sigma^{2n} k \to \cdots$$

colimit is $F_\xi$

$\hat{H}^*(G, F_\xi) \cong \hat{H}^*(G, k) \hat{\xi} \cong H^*(G, k) \hat{\xi}$

Get $E_\xi$ from completing $k \to F_\xi$
to a triangle
\[ \Omega^n \xi \rightarrow k \rightarrow \Omega^n \xi \rightarrow \Omega^n \xi \rightarrow \Omega^n \xi \]
\[ \Omega^{2n} \xi \rightarrow k \rightarrow \Omega^{2n} \xi \rightarrow \Omega^{2n} \xi \rightarrow \Omega^{2n} \xi \]
\[ \Omega^{3n} \xi \rightarrow k \rightarrow \Omega^{3n} \xi \rightarrow \Omega^{3n} \xi \rightarrow \Omega^{3n} \xi \]

\[ \lim \text{ gives } \]

\[ E \xi \rightarrow k \rightarrow F \xi \]
If $V \leq V_\alpha$ is defined by $g_1, \ldots, g_t$ and $W = \{ \text{subvarieties of } V \}$
write $E_V$ for $E_{W}$, $F_V$ for $F_{W}$

$$E_V = E_{g_1} \otimes \cdots \otimes E_{g_t}$$

Complete $E_V \rightarrow k$ to a triangle to get $F_V$.

If $V$ closed hgs irredu = $V_\alpha$
set $W' = \{ \text{subvarieties not containing } V \}$

$$W' = E_V \otimes F_{W'}$$
is an idempotent module corresponding to "the generic point of $V$" — all proper subvarieties have been removed.
For any \( M \) in \( \text{StMod}(kG) \)

\[ \mathcal{V}_G(M) = \{ V \mid M \otimes_k V \text{ not proj} \} \leq \text{Proj } H^*(G, k) \]

- If \( M \) f.g. then \( \mathcal{V}_G(M) \) is the set of subvarieties of \( V_G(M) \)
- \( \mathcal{V}_G(M) = \emptyset \iff M \) is projective
- \( \mathcal{V}_G(M \oplus N) = \mathcal{V}_G(M) \cup \mathcal{V}_G(N) \)
- \( \mathcal{V}_G(M \otimes_k N) = \mathcal{V}_G(M) \cap \mathcal{V}_G(N) \)
- \( \mathcal{V}_G(\mathcal{V}_V) = \{ V \} \)

In particular, every subset of \( \text{Proj } H^*(G, k) \) is \( \mathcal{V}_G(M) \) for some \( M \).
Duality Theorems

Original Poincaré duality spectral sequence: B-Carlson (1994) based on using $L_{q}$'s to make multiple complexes of projective modules.

Easier version to work with:
Greenlees spectral sequence (1995)

$$H^{s,t}_{m} H^{*}(G,k) \Rightarrow H_{s-t}(G,k)$$

Example of consequences:
1. If $H^{*}(G,k)$ is Cohen-Macaulay then it is Gorenstein.
2. If $p$ is a minimal prime then $H^{*}(G,k)_{p}$ is Gorenstein i.e., $H^{*}(G,k)$ is generically Gorenstein
CONSTRUCTION

Choose a h.s.o.p.
\[ k[r_G, \ldots, r_G] \subseteq H^*(G, k) \]

For each \( r_G \), we have a triangle
\[ E_{r_G} \rightarrow k \rightarrow F_{r_G} \rightarrow \Sigma E_{r_G} \]
Truncate to make a complex
\[ \cdots \rightarrow 0 \rightarrow k \rightarrow F_{r_G} \rightarrow 0 \rightarrow \cdots \]
with cohomology \( \Sigma E_{r_G} \) in deg \( 1 \).

Tensor these together:
\[ \Lambda : 0 \rightarrow k \rightarrow \bigoplus_{1 \leq i \leq r} F_{r_G} \rightarrow \cdots \rightarrow \bigotimes_{1 \leq i \leq r} F_{r_G} \rightarrow 0 \]

Cohomology is
\[ \Sigma E_{r_G} \otimes \cdots \otimes \Sigma E_{r_G} \]
\[ \mathcal{V}_G \left( \Omega^r E_{q_1} \bigotimes \cdots \bigotimes \Omega^r E_{q_r} \right) \]
\[ = \mathcal{V}_G \left( \Omega^r E_{q_1} \right) \cap \cdots \cap \mathcal{V}_G \left( \Omega^r E_{q_r} \right) \]
\[ = \bigcap_{i=1}^r \{ \text{subvarieties of } V_k \langle \xi_i \rangle \} \]
\[ = \emptyset \]

so \( \Omega^r E_{q_1} \bigotimes \cdots \bigotimes \Omega^r E_{q_r} \) is projective

Let \( \widehat{P}_* \) be a Tate resolution of \( k \) as a \( kG \)-module. Form a double complex
\[ \hat{E}^{**} = \text{Hom}_{kG} \left( \widehat{P}_*, \Lambda^* \right) \]

Differential from \( \Lambda^* \) first:
\[ E_1 \text{ is } \text{Hom}_{kG} \left( \widehat{P}_*, H^*(\Lambda) \right) \]

so \( E_2 = 0 \)
So the other spectral sequence must also $\Rightarrow 0$.
Differential from $\hat{P}_k$ first:

$$\hat{E}_1^{**} = C^*(\hat{H}^*(G, k) : \Sigma, \ldots, \Sigma_r)$$

(because $\hat{H}^*(G, F_\Sigma) \cong H^*(G, k)_q$)

- stable Koszul complex for computing local cohomology

$$\hat{E}_2^{**} = H^{S, t}_{m*} \hat{H}^*(G, k) \Rightarrow 0$$

Almost, but not quite, Greenlees spectral sequence.
\[ E^{**} = \text{subcomplex of } \hat{E}^{**} \]

consisting of all but the terms \( \hat{E}^{o.t} \) with \( t < 0 \).

\[ 0 \to \text{Tot } E^{**} \to \text{Tot } \hat{E}^{**} \to \text{Hom}_{kG}(\hat{P}^*_*, k) \to 0 \]

\[ H^n \text{Tot } E^{**} \cong H^{n+1} \text{Hom}_{kG}(\hat{P}^*_*, k) \]

\[ \cong H_{-n} (G, k) \]

(Tate duality)

\[ E^{**}_i = C^*(H^*(G, k); \&_1, \ldots, \&_r) \]

\[ E^{s,t}_2 = H_m^{s,t} H^*(G, k) \Rightarrow H_{s-t} (G, k) \]

This is the Greenlees spectral sequence.

Rewrite:

\[ H^{**}_m H^*(G, k) \Rightarrow \text{Im} \]

Remark: Only has nonzero columns between depth & Krull dimension.
Remark

If we remove the whole soc line, not just the negative part, we obtain the Čech complex for $\hat{H}^*(G, k)$:

$$\hat{H}_m^* \cong \hat{H}^*(G, k)$$

Theorem

Let $G$ be a finite group of p-rank $r$ and $k$ be a field of char p. If $H^*(G, k)$ is Cohen-Macaulay, then it is Gorenstein, with canonical module $H^*(G, k)[-r]$. 

Proof

In the Cohen-Macaulay case, the Greenlees spectral sequence has only one column non-vanishing.