Ruchira Datta MSRI Comm. Alg. Intro Workshop Notes

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Introduction to multiplier ideals

Introduction

Setting for all lectures:
let X be a nonsingular affine variety over $\mathbb{C}$,
let $\mathfrak{a} \subseteq \mathbb{C}[X]$ be an ideal
let $c > 0$ be a rational number

Will define a multiplier ideal $\mathcal{J}(\mathfrak{a}^c) \subseteq \mathbb{C}[X]$

$\mathcal{J}(\mathfrak{a}^c)$ measures the singularities of the divisors of $f \in \mathfrak{a}$

("nastier" singularities $\leftrightarrow$ "deeper" multiplier ideals)

$\mathcal{J}(\mathfrak{a}^c)$'s have good formal properties (from Kawamata-Viehweg-Nadel vanishing theorem)

Three Approaches

Algebraic (Lipman) called them adjoint ideals

Analytic (Nadel, Demailly, Siu)

Geometric (Esnault-Viehweg, Kawamata, Ein, ...)
will focus on this

Lazarsfeld A

9/10/2002
① b of 3b

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Plan

Lecture A: Definition & Examples

Lecture B: Invariants, Skoda's theorem, Artin-Rees

Lecture C: Asymptotic constructions, application to symbolic powers

Log Resolution of $\mathfrak{a}$

idea: blow up the ideal so it becomes locally principal - generated by a single monomial

Def A log resolution of $\mathfrak{a}$ is a projective birational map $\mu: X' \to X$ with X' nonsingular, $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$, F effective divisor s.t. F + (exceptional divisor of μ) has simple normal crossing support (local analytically: $z_1^{a_1} \cdots z_k^{a_k} = 0$)

Example $\mathfrak{a} = (s^2, t^2) \subseteq \mathbb{C}[s,t]$

$X' = \underbrace{Bl_0}_{\text{blowup}}(\mathbb{C}^2) \to \mathbb{C}^2 = X$

locally $(u,v) \longmapsto (u, uv)$ ($(u=0)$ is exceptional E)

$\mathfrak{a}\,\mathcal{O}_{X'} = (u^2, (uv)^2) = u^2(1, v^2) = u^2$

$\Rightarrow \mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-2E)$ here $F = 2E$

Example 2 $\mathfrak{a} = (s^3, t^2)$

Get log resolution in 3 steps:

Step 1: $X_1 = Bl_0(\mathbb{C}^2) \longrightarrow \mathbb{C}^2$

$(u,v) \longmapsto (u, uv)$

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$\mathfrak{a}\cdot\mathcal{O}_{X'} = (u^3, u^2v^2) = u^2(u, v^2)$

... see transparency

$\mathfrak{a}\cdot\mathcal{O}_{X'} = \mathcal{O}(-2E_1 - 3E_2 - 6E_3)$

$K_{X'/X} = E_1 + 2E_2 + 4E_3$

Hironaka showed: Log resolutions exist

Notation: $\mathfrak{a}\cdot\mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$

Write $F = \sum r_i E_i$ (E_i prime divisors)

<u>Def</u> Given $\mu : X' \to X$,

$K_{X'/X} = K_{X'} - \mu^* K_X$ relative canonical bundle

$= \det(d\mu)$

<u>Notation</u> $K_{X'/X} = \sum b_i E_i$ $(b_i \geq 0)$

Multiplier Ideals

Given $\mathfrak{a} \subseteq \mathbb{C}[X]$, construct log resolution $\mu : X' \to X$, $\mathfrak{a}\cdot\mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$

<u>Def</u> (Lipman)

$$\mathcal{J}(\mathfrak{a}) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - F)$$

$$= \{h \in \mathbb{C}[X] \mid \operatorname{div}(\mu^* h) + K_{X'/X} - F \geq 0\}$$

$$= \{h \in \mathbb{C}[X] \mid \operatorname{ord}_{E_i}(h) \geq r_i - b_i \ \forall i\}$$

$-F$ will be ample; canonical + ample = adjoint

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<u>Homework</u>: $\bar{\mathfrak{a}} \subseteq \mathcal{J}(\mathfrak{a})$

<u>Bring in the coefficient c</u>:

Suppose $c \in \mathbb{N}$. Replace $\mathfrak{a}$ by $\mathfrak{a}^c$. Then μ is still a log resolution of $\mathfrak{a}^c$, except now $\mathfrak{a}^c \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-cF)$

So

(*) $\mathcal{J}(\mathfrak{a}^c) = \{h \in \mathbb{C}[x] \mid \mathrm{ord}_{E_i}(h) \geq [cr_i] - b_i \text{ for all } i\}$

<u>Def</u> For <u>any</u> $c > 0$,

$$\mathcal{J}(\mathfrak{a}^c) = \text{ideal defined by } (*) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - [cF])$$

<u>Thm A</u> This definition is independent of the resolution μ.

(HW: $\mu_* \mathcal{O}_{X'}(-[cF])$ is <u>not</u> independent of the resolution)

<u>Thm B</u> (Special case of Kawamata-Viehweg vanishing theorem)
Have $R^j \mu_* \mathcal{O}_{X'}(K_{X'/X} - [cF]) = 0$ for $j > 0$
(so expect $\mathcal{J}(\mathfrak{a}^c) = \mu_*(K_{X'/X} - [cF])$ to have particularly good properties)

<u>Example 1</u>

when exponent < 0, the whole ring (by convention)

$\mathfrak{a} = (s^2, t^2)$ $\quad \mathcal{J}(\mathfrak{a}^c) = (s,t)^{[2c]-1}$

Especially notice: $\mathcal{J}(\mathfrak{a}^c)$ is nontrivial $\Leftrightarrow c \geq 1$

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<u>Example 2</u> $\mathfrak{a} = (s^3, t^2)$;

$\mathcal{J}(\mathfrak{a}^{5/6}) = (s, t)$

(i.e., $\underset{\text{cusp}}{\prec}$ has worse singularity than $\underset{\text{crossing}}{X}$)

Analytic construction:

Take generators $\mathfrak{a} = (g_1, \dots, g_p)$

$$\mathcal{J}(\mathfrak{a}^c)^{an} = \left\{ \text{holomorphic } h \;\middle|\; \frac{|h|^2}{\sum |g_i|^2} \text{ is locally integrable} \right\}$$

(For $h \in \mathbb{C}\{z_1, \dots, z_d\}$,
$$\int \frac{|h|^2}{\prod |z_i|^{2a_i}} < \infty \iff z_1^{\lfloor a_1 \rfloor} \cdots z_d^{\lfloor a_d \rfloor} \,|\, h$$)

Monomial ideals

$\mathfrak{a} \subseteq \mathbb{C}[t_1, \dots, t_d]$ a monomial ideal

identify monomial with its vector of exponents
$v \in \mathbb{N}^d \subset \mathbb{R}^d$
$v \in \mathbb{N}^d \longleftrightarrow x^v$

polytope $P(\mathfrak{a})$ = convex hull of all exponents v s.t. $x^v \in \mathfrak{a}$

write $\mathbb{1} = (1, \dots, 1)$

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Thm (Jason Howald)
$\mathcal{J}(\mathfrak{a}^c)$ = monomial ideal spanned by all x^v
s.t. $v + \mathbb{1} \in \text{int}\,(cP(\mathfrak{a}))$

<u>Example</u> $\mathfrak{a} = (t_1^{m_1}, \ldots, t_d^{m_d})$

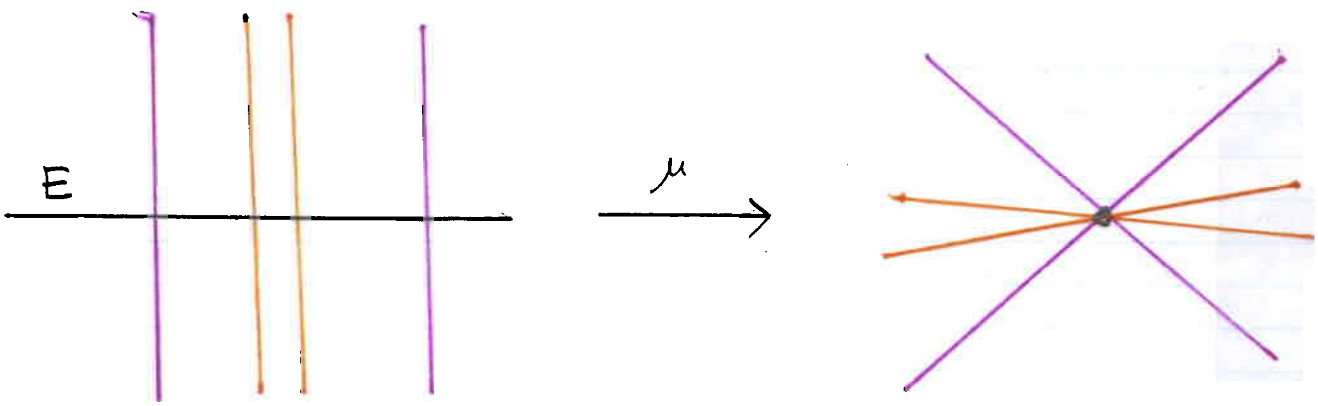
$\mathcal{J}(\mathfrak{a}^c) = \langle t_1^{e_1} \cdots t_d^{e_d} \mid \sum \frac{e_i + 1}{m_i} > c \rangle$

<u>Esp</u> $\mathcal{J}(\mathfrak{a}^c)$ nontrivial $\Leftrightarrow c \geq \sum 1/m_i$

Open Question:
Given $q = \bar{q}$, is $q = \mathcal{J}(\mathfrak{a}^c)$ for some $\mathfrak{a}$ & c?

candidate: lines through origin in $\mathbb{C}^3$ 

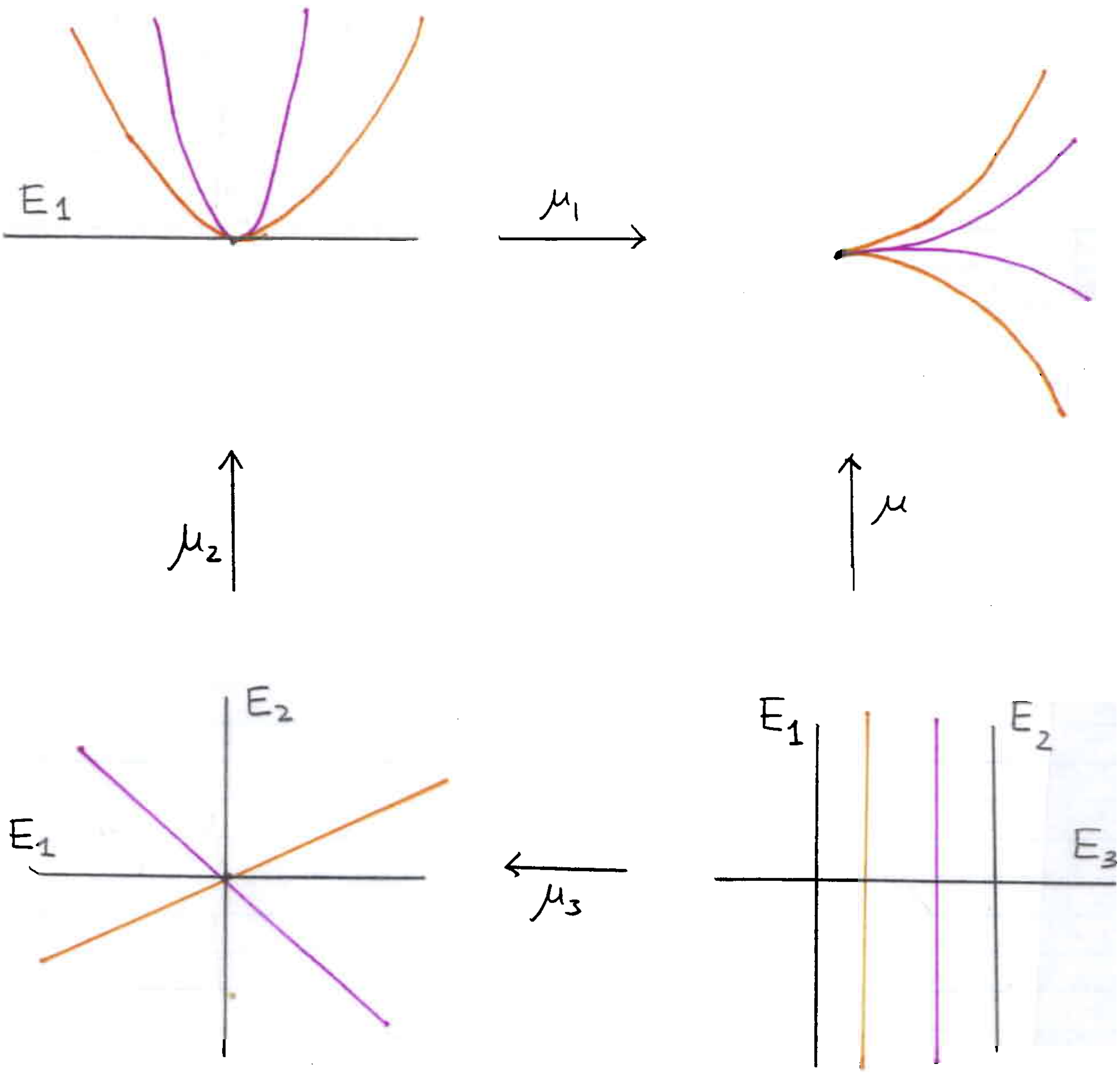
Log Resolution of $\mathfrak{a} = (s^2, t^2)$



<u>Pictured:</u> $s^2 - t^2 = 0$
$5s^2 - t^2 = 0$

$$\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-2E)$$

Log. Resoln of $\mathfrak{a} = (s^3, t^2)$



$$\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}(-2E_1 - 3E_2 - 6E_3)$$

$$K_{X'/X} = E_1 + 2E_2 + 4E_3$$

Computation of $\mathcal{J}\left((s^3,t^2)^{5/6}\right)$:

$\mu: X' \longrightarrow \mathbb{C}^2$ as before, $(s^3,t^2)\cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$.

Need: $\mu_* \mathcal{O}_{X'}\left(K_{X'/X} - [\tfrac{5}{6}F]\right)$

$$K_{X'/X} = E_1 + 2E_2 + 4E_3$$

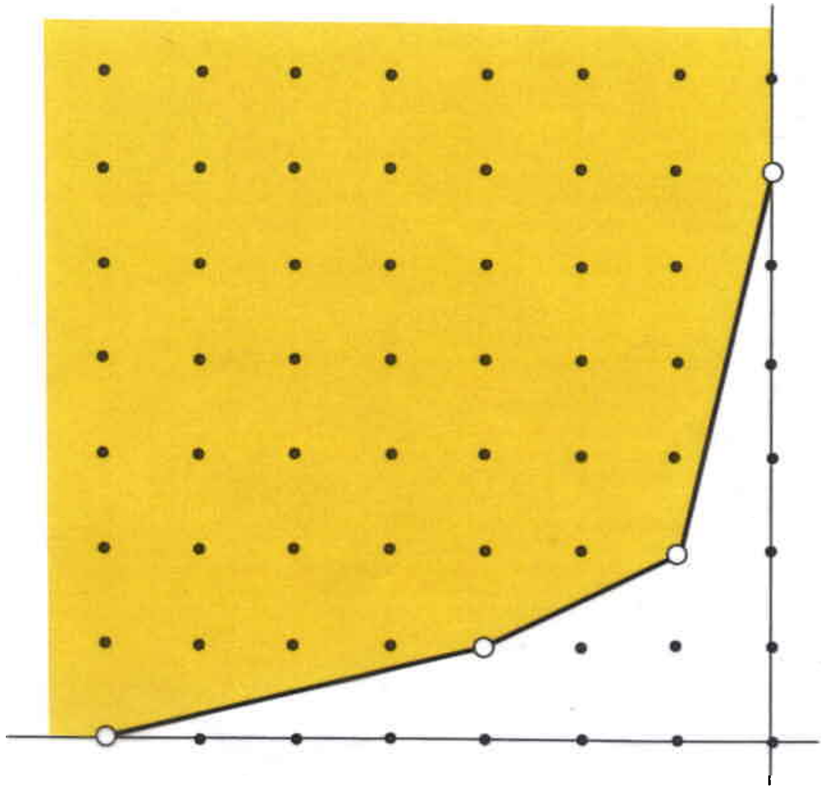
$$F = 2E_1 + 3E_2 + 6E_3$$

$$\tfrac{5}{6}F = \tfrac{10}{6}E_1 + \tfrac{15}{6}E_2 + 5E_3$$

$$\left[\tfrac{5}{6}F\right] = E_1 + 2E_2 + 5E_3$$

$$K_{X'/X} - \left[\tfrac{5}{6}F\right] = -E_3$$

$$\mathcal{J}\left((s^3,t^2)^{5/6}\right) = \mu_* \mathcal{O}_{X'}(-E_3) = (s,t)$$



$$\mathfrak{a} = (y^6, xy^2, x^3y, x^7)$$

$$\mathcal{J}(\mathfrak{a}) = (y^2, xy, x^3)$$

$$\left(\text{If } c<1,\ y,\ x^2 \in \mathcal{J}(\mathfrak{a}^c)\right)$$