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The geometry of syzygies

roots in invariant theory - Hilbert

let \( S = K[x_0, \ldots, x_r], m = (x_0, \ldots, x_r) \)

invariant ring

\( S^G = K[f_1, \ldots, f_t] \leftarrow K[y_1, \ldots, y_t] \)

Hilbert wrote formulas counting these

variety

\( X \subseteq \mathbb{P}^r \quad \mapsto \quad S_X = S/I_x, \text{its coord ring} \)

family of varieties

\( X \subseteq \mathbb{P}^r \times \mathbb{A}^s \quad \mapsto \quad S_{(X)} = S[I_{t_1, \ldots, t_s}]/I \)

\( X \mapsto S_X \quad I_t = I(p) \)

\( S_X \mapsto \text{Hilbert function} \quad H_{S_X}(d) = \dim_K(S_X)_d \)

\( \text{Thm } H_{S_X}(d) \overset{=} \rightarrow P_X(d), \text{a polynomial}(= \chi_X(d)) \)

\( X \text{ is flat} \iff P_{X_p}(d) \text{ is identically } = P(d) \)

(\( \text{independent of } p \))

\( \text{Thm } \text{The family } S_{X_p} \text{ is flat} \iff H_{S_X}(d) \text{ is constant} \)

\( \text{Thm } \text{Let } M = \bigoplus M_d \text{ be a graded } S \text{-module} \)

Then \( H_M(d) = \dim_K M_d \text{ is polynomial for large } d \), and more...

\( \text{Proof (Hilbert) Let } S(-a) \text{ be the rank 1 free module with generator in degree } a. \text{ (This makes the formula } M(b)_d = M_{b+ad} \text{ work out.)} \)
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Looking at the combinatorics shows

\[ H_2(d) = \binom{r+d}{r} \quad \text{a polynomial for } d \geq r \]

Then \( H_2(a) = \binom{r+d-a}{r} \)

\[ 0 \to \oplus S(-a_j) \to \cdots \to \oplus S(-a_j) \to M \to 0 \]

\[ F_t \to \cdots \to F_1 \to F_0 \text{-free modules} \]

Hilbert Syzygy Theorem says: \( t \leq r+1 \)

Corollary \( H_2(d) = \frac{1}{2} \sum_{i=0}^{t} (-1)^i \sum_j \binom{r+d-a_j}{r} \)

Write \( F_i = \oplus S(-j)^{\beta_{ij}} \) (a different \( j \))

\( i.e., \beta_{ij} \text{ copies of the degree } j \text{ piece} \)

\( \{ \beta_{ij} \} \text{ are the "graded Betti numbers"} \)

\[ H_2(d) = \frac{1}{2} \sum_{i=0}^{t} (-1)^i \sum_j \beta_{ij} \binom{r+d-j}{r} \]

Example: \( X = 4 \text{ points in } \mathbb{P}^2 \)

\[ X = \mathbb{P}^2 \times (\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \text{-diagonals}) \to T \]

for basis of dim 4, (in each graded piece)

take high degree curve going through

3 pts & not 4th:

* nonzero on one pt,
  zero on others
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9/11/2002
②a of 3b

So \( P_x^s(d) = 4 \)

no 3 on a line - then up to \( \text{PGL}(4) \),
the 4 pts are
\[
(1,0,0) \quad (0,1,0) \quad (1,1,1) \quad (0,0,1)
\]

\( X = 4 \text{pts} \subset \mathbb{P}^2 \)

Cases: 1) No 3 on a line

\[
\begin{array}{cccc}
H_x(d) & 0 & 1 & 2 & 3 \\
1 & 3 & 4 & 4 & 4
\end{array}
\]

\( \bigcirc \) 2 conics

or conics as pairs of lines

\( \bigcirc \)

how many cubic forms
are there? \( \binom{2+3}{2} = 10 \)

could get cubic forms as products of linear forms & quadratic forms

\( \ell_1 Q_1 = \ell_2 Q_2 ? \) do these overlap?

- then would have common divisor

\[
0 \rightarrow S(-4) \rightarrow S(-2)^2 \rightarrow S \rightarrow S_x \rightarrow 0
\]

\[
\begin{pmatrix}
Q_2 \\
Q_1
\end{pmatrix}
\]

\( \beta_{24} = 1 \)
\( \beta_{12} = 2 \)
\( \beta_{02} = 1 \)

the rest of the Betti numbers are zero

Betti diagram

\[
\begin{array}{cccc}
\beta_{00} & 0 & 1 & - \\
\beta_{12} & 2 & - & - \\
\beta_{24} & - & 1 & -
\end{array}
\]
A digression: Minimality

A complex of free modules \( \cdots \rightarrow F_{i+1} \xrightarrow{\phi_{i+1}} F_i \xrightarrow{\phi_i} F_{i-1} \) is minimal if \( \phi_i(F_i) \subseteq m F_{i-1} \).

In general, the Betti diagram looks like:

\[
\begin{array}{c|cccc}
   & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & \beta_0 & \beta_1 & \beta_2 & \beta_3 \\
1 & & & & \\
2 & & & & \\
3 & & & & \\
\end{array}
\]

Cases

3 points on a line, one off:

\[
\begin{array}{c|cccc}
   & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 1 & 2 & 3 & 4 \\
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

Quadratic function vanishing on

3 pts but not off pt: product of linear forms

2 pts & off pt, but not other on pt:

\[
\begin{array}{c|cccc}
   & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 1 & 2 & 3 & 4 & 4 \\
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

4 points on a line

\[
\begin{array}{c|cccc}
   & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 1 & 2 & 3 & 4 & 4 \\
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

A quadric containing 3 collinear pts also contains the 4th.
back to:
3 pts on a line, one off
quadrics vanishing on all 4 pts
products of linear forms $A$
$A, B_1, AB_2$
have syzygy:
$B_2(AB_1) = B_1(AB_2)$
$\Rightarrow$ these don't give all cubics,
need one more to generate:

$$C = F_1 B_1 + F_2 B_2$$

$$0 \rightarrow S(-3) \rightarrow S^{2(-2)}(AB_1, AB_2, C) \rightarrow S(-3)$$

Betti diagram:

$$\begin{array}{c}
1 \\
2 & 1 \\
1 & 1 \\
\end{array}$$

Hilbert function may be written as generating function
$$\sum H_m(d)t^d = \frac{1}{(1-t)^r}$$
Betti diagram for 4 pts on a line