Puchira Datta MSRI Comm. Alg. Workshop Notes  
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Commutative algebra of N points in the plane

Notes for these lectures are at  
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Suppose $P_1, \ldots, P_n \in \mathbb{C}^2$. Let $E = \mathbb{C}^{2n}$  
$C[E] = \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$.  

**Coincidence locus** $V_{ij} = V(x_i - x_j, y_i - y_j)$

$V = \bigcup_{i \neq j} V_{ij}, \quad I(V) = \bigcap_{i < j} (x_i - x_j, y_i - y_j)$

**Problem:** Describe $I(V)$.

Case $P_1, \ldots, P_n \in \mathbb{C}^1$ points on a line  
$J = \bigcap_{i \neq j} (x_i - x_j)$

**Observes:**

1) $J = (\Delta(x))$  
   where $\Delta(x) = \prod_{i < j} (x_i - x_j) = \det \begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{bmatrix}$  

2) $J$ is a free $C[x]$-module

3) $J^m = (\Delta(x))^m = \bigcap_{i < j} (x_i - x_j)^m = J(m)$

4) Rees algebra $C[x][tJ]$ is a polynomial ring.

all follows because $V(J)$ is a hyperplane arrangement.

higher dim: more general subspace arrangement
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Now what happens on plane?
Consider (1)
\( D = \mathbb{N} \times \mathbb{N} \quad D_1 = (x_1, y_1, x_1, y_1, \ldots, x_n, y_n) \)
\( D_0(x, y) = \det \begin{bmatrix} x_1 & y_1 & \cdots & x_n & y_n \\ x_1 & y_1 & \cdots & x_n & y_n \end{bmatrix} \)
\( \sigma \in S_n, \quad \sigma x_i = x_{\sigma(i)}, \quad \sigma y_i = y_{\sigma(i)} \)
\( \sigma D_0(x, y) = \text{sgn} (\sigma) D_0 \)
\( \Rightarrow D_0 \in \mathbb{C}[x, y]^E \text{ space of alternating polynomials} \)

In fact, \( \{ D_0 \} : \text{all } D_0 \text{ is a vector space basis for } \mathbb{C}[x, y]^E \)

Theorem 1 \( I = (D_0 : \text{all } D_0) \)

minimal generators?

\( I \) is doubly graded by degree in \( x \) & degree in \( y \)

\( M = I / (x, y) I \)

Theorem 2 \( \dim M = C_n = \frac{1}{n+1} \binom{2n}{n} \), the \( n \)th Catalan #

\( M = \bigoplus_{r,s} M_{r,s} \quad C_n = \# \lambda \leq S, \quad S = \{ (n-1, n-2, \ldots, 1) \}
\)

\( a(\lambda) = |S| - |\lambda| \)
\( b(\lambda) = \# \text{ hooks in } \lambda \)

\( s.t. \ a \in \text{ leg of arm} \)

hook: pick a square on border vertically above it is its leg horizontally right of it is its arm

Theorem (Garsia- Haglund- Haiman)
\[ \sum_{r,s} q^{r/s} \dim M_{r,s} = \sum_{\lambda \in S} q^{a(\lambda)} + b(\lambda) \]
Problem: Find $D_\lambda$ for each $\lambda \leq S$

\[ \text{s.t. } \deg_x \Delta D_\lambda = a(\lambda), \deg_y \Delta D_\lambda = b(\lambda) \]

\[ I = (D_\lambda : \lambda \leq S) \]

\[ n = 3 \]

\[ \begin{array}{c|c|c|c}
0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 2 \\
1 & 1 & 2 & 3 \\
2 & 2 & 0 & 1 \\
3 & 3 & 1 & 0 \\
\end{array} \]

Consider (2):

Theorem 4: $I$ is a free $C[x]$-module

(Hence a free $C[x, y_1]$-module.)

depth $I = n+1$.

Consider (3):

Theorem 5: $I^m = \bigcap_{i,j} (x_i - x_j, y_i - y_j)^m$ for all $m$.

Theorems 1, 4, and 5 follow from

Theorem 6: $\forall m (\Delta_0 : \text{all } D)^m$ is a free $C[x]$-module.

Corollary: $(\Delta_0 : \text{all } D)^m = I(m)$

Pf of Cor from Thm 5

Step 0: $I$ is clear.

Step 1: localize at $P$ where $P_i \neq P_j$, both sides factor. So $(\Delta_0)^m_P = I_P^m$ by induction on $n$.

Step 2: $\exists (\Delta_0)^m_P$, has depth $\geq n-1$ as a $C[x]$-module.

\[ \Rightarrow \text{no associated prime supported in } V(x_1 - x_2, \ldots, x_{n-1} - x_n) \]
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\(f \in (\Delta_0)^m \quad \text{for} \quad \Delta_0 \subseteq \text{Spec} \mathbb{C}[x] \Rightarrow f \in (\Delta_0)^m\)
(Holds for \(f \in I^m\) by Step 1.) \(\Box\)

Consider (4)
Rees algebra \(R = \mathbb{C}[x,y][t(\Delta_0)]\)
\[X = \text{Proj } R \quad X \rightarrow E \quad \downarrow \]
\[X/5_n \rightarrow E/5_n \]

Prop \(X/5_n = \text{Hilb}^n(\mathbb{C}^2)\)
- the Hilbert scheme of points in the plane
\[= \{ I \subseteq \mathbb{C}[x,y] : \text{dim}_\mathbb{C} \mathbb{C}[x,y]/I = n \} \]

Jhm (Fogarty, 60's) \(\text{Hilb}^n(\mathbb{C}^2)\) is irreducible & nonsingular.

true only of \(\mathbb{C}^2\), makes things work out nicely

Remark codim "y-axis" in \(\text{Hilb}^n(\mathbb{C}^2)\) is \(n\)
In particular, \(\text{dim } R/(y) = n+1 = \text{dim } R/x\)

Jhm7F is Gorenstein \((X \text{ is arithmetically Gorenstein})\)
This implies Theorem 6
however, the existing proof assumes Theorem 6

Problems:

Prove Jhm 6 and/or Jhm 7 directly

7. Improve Jhm 7. Does \(R\) have rational singularities?

\(V = \bigcup V(x_i - x_j)\) is a hyperplane arrangement in \(\mathbb{C}^2\)
\(V = \mathbb{C}^2 \ominus V\)
\(x_i = \sum \mathbb{C}^2 \ominus V(x_i - x_j) \subseteq \mathbb{C}^2 \ominus V\)
What about \(i(x) I = I(\mathbb{C}^2 \ominus V)\)?
Also for other hyperplane arrangements?
(must be a free hyperplane arrangement)