Teissier 3
Monomial ideals, binomial ideals, polynomial ideals. 

REMORES:\n\[ Z(\mathbb{R}) \xrightarrow{\pi(\mathbb{R})} \mathbb{A}^d \]
\[ \mathcal{Y} \xrightarrow{\mathcal{T}} \mathcal{X} \]

Is \( \mathcal{T} \) an isomorphism outside \( \text{Sing} \mathcal{X} \)?

X is irreducible binomial variety, \( X \subseteq \mathbb{A}^d(\mathbb{K}) \); \( k = \overline{k} \)

THM (Gonzalez-Perez) \( X \) is non degenerate and \( \mathcal{T} \) is an isomorphism outside \( \text{Sing} \mathcal{X} \).

\((\lambda^m - \lambda^m \mu^s)\) prime ideal. \( 1 \leq s \leq d \)

This ideal satisfies the condition of THM in Talk 2.

How about dim of the ideal?

Eul. - Stur.
\[ \mathcal{L} = \langle \mathcal{M}^s - \mathcal{M}^s \rangle \leq \mathbb{Z}^d \]

\( X \) is irreducible. \( \iff \mathcal{L} \) is a direct factor in \( \mathbb{Z}^d \)

\[ x_{\mathcal{L}} \cdot \text{coker} \mathcal{X} \]

Let \( c = x_{\mathcal{L}} \). \( L' \subset L \) \( \| L' \| = c \)

\[ u_{i_1} \ldots u_{i_k} T_{i_1, L'} = \prod_{L \in L'} \frac{\lambda^m}{\det(\langle \mathcal{M}^s - \mathcal{M}^s \rangle)} \]

Let \( \lambda^m - \lambda^m \mu^s + \nu \mu^p \), suppose that \( \langle a'_{i}, p \rangle > \langle a'_{i}, m \rangle \)

for all \( i \). This means that, after the transformation in the \( y's \), we get:

\[ y'(y' - \lambda^m + \nu y') \]
The goal is to do some kind of deformation into bimomial ideals, and then the map the resolves, the bimomial variety works in general.

Consider:

A 1-dimensional complete noetherian integral domain, algebraically closed residue field, equal characteristic.

\[ k[[u_1, u_2]] / \mathfrak{f}(u_1, u_2) = R \]

\[ R \cong k[[t]] : = \text{integral closure of } R \text{ in its quotient field (finite } R\text{-module)} \]

\[ \exists \text{ a conductor element, i.e. } f \in R \text{ s.t. } h \in k[[t]] \backslash R \]

\[ H : = \text{values of } t\text{-adic valuation on } R \in \Gamma N. \]

\[ \Gamma = \langle \delta_1, \ldots, \delta_{g+1} \rangle \text{ where } \delta_{i+1} \text{ is not in the subgroup generated by } \langle \delta_1, \ldots, \delta_i \rangle. \]

but

\[
\begin{align*}
\{ \text{relations } \} & \quad \text{where } k = 1, \ldots, g \\
\text{other relations.}
\end{align*}
\]

consider the curve \[ C^h : = \{ u_1 = \xi_1^\ast, \ldots, u_{g+1} = \xi_{g+1}^\ast \} \]

thus curve \[ C^h \text{ is defined by a finite number of bimomial } g \text{ relations. It is also true that } C^h \text{ is irreducible curve.} \]

Therefore we can build a fan and get:

\[
\begin{array}{c}
Z(\mathcal{X}) \ar[r] & \mathbb{A}^{g+1} \\
\downarrow & \ar[u] \\
W^h & \ar[r] & C^h
\end{array}
\]

What is the relation between the two curves?
By definition:
For each \( i, \, 1 \leq i \leq g + 1 \) there exists \( a \in \mathbb{F}_i(\mathfrak{t}) \in \mathbb{R} \) \( \mathfrak{f}_i = t^{x_i} + \sum_{j \neq i} c_j(\mathfrak{t}) t^j \)

divide by \( \mathfrak{f}(\mathfrak{t}) \mathfrak{f}(\mathfrak{t}) \), get
\( \mathfrak{f}(\mathfrak{t})^{-1} = t^{x_i} + \sum c_j(\mathfrak{t}) t^{j-x_i} \)

This is a parametricization from the curve to the monomial curve (\( n = \frac{1}{2} \)) while \( n \to 0 \) gives a curve isomorphic to the original. I want a system of equations as: \( u^m \cdot u^{m_2} \cdot \sum c_{p} (\mathfrak{t}) u^{p} \)

A different approach:

\[ R \subseteq V \] where \( V \) is a valuation ring.

In \( V \):
\( a, b \in V \) \( \iff a \mid b \)
\( a \mid b \iff a \mid b \) and \( b \mid a \)

\( V \setminus 0 = \mathbb{Q}_+ \cup 0 \) positive part of a totally ordered group \( \mathbb{Q}_+ \)

\( k^* \xrightarrow{\psi} \mathbb{Q}_+ \)

Multiplicative group of \( k \)
\( k = \text{tot } V \).

This map has the following property:
\( \psi(x) = x \) modulo \( \psi \)
\( \psi(xy) = \psi(x) + \psi(y) \)
\( \psi(m + \psi) \geq \min(\psi(x), \psi(y)) \)

\( \mathbb{P}_\psi^+(R) = \{ x \in R \text{ such that } \psi(x) \geq 0 \} \)
\( \mathbb{P}_\psi^-(R) = \{ x \in R \text{ such that } \psi(x) > 0 \} \)
\( \oplus P_\phi(R) / P_\psi(R) \cong k[(u_m)] / <u_m - u_m^m> \)

Compute \( g_{\psi}(R) = C[[t^{e_1}, \ldots, t^{e_{n+1}}]] \) c \( C[[t]] = g_{\alpha}(\varepsilon) k[[t]] \).

If \( R \cap \Gamma_m = \emptyset \) then \( \forall m \in \mathbb{N} \).

\( R_{\psi}(R) = \bigoplus_{m \in \mathbb{N}} R_{m} \cdot u^{-m} \subset R[[u, u^{-1}]] \) say \( R_{m} \subset R \) when \( m \leq 0 \).

In the special case when \( \Gamma_m = \text{I}_m \) then \( R_{\psi}(R) = R[[u, u^{-1}]] \subset R[[u, u^{-1}]] \).

Since \( R \) is equicharacteristic, \( k \subset R \).

\( k[[u]] \rightarrow R_{\psi}(R) \)

So there is a map \( \text{Spec}(R_{\psi}(R)) \rightarrow \text{Spec}(k[[u]]) \) flat (faithfully).

There is also a map:

\( R_{\psi}(R) \rightarrow g_{\psi}(R) \)

\( x \cdot u^{-m} \rightarrow \text{im}_{\psi}(x) \in P_{\psi}(x_{m+1}) \)

\( x \cdot u^{-m} = (xv)u^{-m+1} \)

So the kernel is generated by \( u = x \).

Moreover:

\( x_m u^{-m} \rightarrow x_m u^{-m} \in R \)

The kernel is \( (v - c)R_{\psi}(R) \).
Map \( k[u_1, \ldots, u_{g+1}] \rightarrow R \)
\[ u_i \mapsto \delta_i \]

Get a valuation \( \lambda \) in \( R \) and a filtration on \( R \):
\[ \text{gr}_\lambda k[u] = k[u] \rightarrow \text{gr}_\lambda R \]
\[(u_{m_1}, u_{m_2}) \subset k[u] \rightarrow \text{gr}_\lambda R \]

Since we were looking for a system of equations of the type:
\[ u_{m_1}^s - u_{m_2}^{s_1} + \sum \lambda_i^p(s)(w)u_i = 0 \]

We need \( \text{weight}(u_i^p) > \text{weight}(u_{m_1}^s) = \text{weight}(u_{m_2}^{s_1}) \).

Otherwise said, we want:
\[ \langle p, a_i \rangle \geq \langle m_1, a_i \rangle \quad \forall i. \]

\textbf{Example:} The simplest degenerate planar curve:
\[ (V_1^2 - V_0^2)^2 - V_0 V_1 = 0 \]
\[ \Pi = \langle 1, 2, 1(3) \rangle. \]
The graded ring
\[ u_1^2 - u_3^2 = 0 \]
\[ u_2^3 - u_0^5 u_1 = 0 \]

Parameter \( \nu \):
\[ u_1^2 - u_0^3 - \nu u_2 = 0 \]
\[ u_2^3 - u_0^5 u_1 = 0 \]

\[ \delta \]