CRAIG HUNEKE (p. 1) 2-3 9:30 AM

Auslander-Reiten Question: Let (R, m) be a local Noeth. comm. ring. M a f.g. R-module. If $\mathrm{Ext}^i_R(M, M \oplus R) = 0$ $\forall i > 0$, then M is free.

Theorem (Hartshorne) R is a graded domain over a perfect field k of char p. R isolated non-CM point at the vertex. Then R has a f.g. maximal CM module.

PROOF

Let $q = p^e$. Consider $R^{1/q}$ as R-module.

$H^i_m(R)$ fin. length $0 \le i < \dim R$

$\boxed{H^i_m(R^{1/q})} \simeq H^i_m(R)$

f. length

$$R^{1/q} \hookleftarrow R \qquad R \hookleftarrow R^q \qquad \Big| \qquad R^{1/q} = \bigoplus_{\alpha \in \mathbb{Z}/q\mathbb{Z}} M_\alpha$$

$\Rightarrow$ for $q \gg 0$, $\exists M_\alpha \mid R^{1/q}$ s.t. $H^i_m(M_\alpha) = 0$, $i < \dim R$.

A CM local ring has finite CM type if $\exists$ only f. m. iso. classes of indec. MCM.

- Aus-Reit. Conjecture is true if R has finite CM type

Theorem (Auslander) R is a complete local ring, CM, having finite CM type, then R has an isolated singularity.

PROOF (Leuschke - H.)

R has an iso. sing $\Leftrightarrow$ $\mathrm{pd}_{R_p} k(p) < \infty$ $\forall$ primes $p \neq m$ ($\dim R/p = 1$ enough, WOLOG)

$\Leftrightarrow$ the free resolution of R/p over R splits at all primes $Q \neq m$ eventually. $\Leftarrow$ $\forall$ MCM's M, N, $\underline{\mathrm{Ext}^1_R(M,N)}$ has finite length.

(CH p2)

$\alpha \in \mathrm{Ext}^1(M,N)$

$r_1, \dots, r_n \in \mathfrak{m}_R$

$$\begin{tikzcd}
\alpha := 0 \arrow[r] & N \arrow[r] \arrow[d, "\cdot r_1"] & X \arrow[r] & M \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[d, "\cdot r_2"] & X_1 \arrow[r] & M \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r] & X_2 \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & & \vdots & &
\end{tikzcd}$$

$\exists\, i,j \quad i \neq j$ s.t. $X_i \cong X_j$

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "r_{i+1}\cdots r_j"'] & X_i \arrow[r] & M \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r] & X_j \arrow[r] & M \arrow[r] & 0
\end{tikzcd} \qquad (j > i)$$

$0 \to N \to N \oplus X_i \to X_j \to 0$: Middle $\cong N \oplus X_j$

Miyata's Thm $\Rightarrow$ this splits $\Rightarrow$ $0 \to N \to X_i \to M \to 0$ splits //

PAST RESULTS

② Auslander-Ding-Solberg ('93) If R is a complete intersection, ($R = {}^{RLR}/_{\text{reg seq}}$ or $\hat{R}$ = same) and $\mathrm{Ext}_R^2(M,M) = 0 \Rightarrow \mathrm{pd}_R M < \infty$.

If $\mathrm{Ext}^i(M,R) = 0 \ \forall i > 0$, then M is free.

③ Avramov-Buchweitz Same hypothesis on R, same conclusion if $\mathrm{Ext}_R^{2i}(M,M) = 0$ for some $i \geq 1$. ('00)

④ L. Sega, A. Vraciu, —) If R is Artinian comm., $\mathfrak{m}^2 M = 0$, then the conj. is true. (In particular if $\mathfrak{m}^3 = 0$.)

(CH p3)

Theorem (Leuschke, -H.) If R is CM, local, $R \cong S/(\text{reg seq})$, ω_R exists, S is either Gorenstein, normal or S is CM normal and contains $\mathbb{Q}$, then Aus-Reit question has a positive answer.

$*$ Not all Artinian rings are $\frac{\text{CM normal}}{\text{reg. seq.}}$; e.g. $R = \frac{k[X_1, X_2, X_3, X_4]}{(7 \text{ general quadrics})}$

Reductions

1. WOLOG $R = S$ (Aus-Ding-Sol)
2. Replace M by $syz_i(M)$; WOLOG M is MCM, reflexive
3. WOLOG $Ext^i_R(M^*, R) = 0$ $\forall\, 1 \le i \le D$ choose D as you wish
   $M^* = Hom_R(M, R)$

$$0 \to M' \to F_D \to \cdots \to F_0 \to M \to 0$$
$$0 \to M^* \to F_0^* \to \cdots \to F_D^* \to (M')^* \to 0 \quad \text{exact}$$

$$M^* \otimes M \xrightarrow{\theta} Hom_R(M, M) \qquad M \text{ free} \Leftrightarrow \theta \text{ is an } \cong$$
$$f \otimes x \longmapsto [y \longmapsto f(y)x]$$

R normal, M MCM, M_p is free if $ht\, p \le 1$. So θ is an $\cong$ in codim ≤ 1.

STRATEGY: Prove $M^* \otimes M$ is CM

PROVE: $\omega \otimes M^* \otimes M$ is CM

Lemma: Suppose R_p is Gor. $\forall\, ht\, p = 0$, $\exists\, \omega_R$, N is a MCM with rank. If $Ext^i_R(N, R) = 0$, $1 \le i \le \dim R$, then $N \otimes \omega$ is MCM.

Lemma Suppose M is a MCM with rank. Assume R CM
(1) $Hom_R(M, M)$ is CM
(2) $\exists$ s.o.p. $\underline{x}$ s.t. with $\overline{R} = R/\underline{x}$, $\overline{Hom_R(M, M)} \cong Hom_{\overline{R}}(\overline{M}, \overline{M})$

(CH p.4)

Then $M \otimes M^\vee$ is MCM, $M^\vee = \mathrm{Hom}(M, \omega)$.

<u>Claim</u> $M^\vee \simeq M^* \otimes \omega$

If $\boxed{M^* \otimes \omega \text{ is CM}}$, then ETS $M^{\vee\vee} \simeq (M^* \otimes \omega)^\vee \simeq \mathrm{Hom}(M^* \otimes \omega, \omega)$

$\simeq \mathrm{Hom}(M^*, R) = M^{**} = M$

By 1st lemma + ③, $M^* \otimes \omega$ is CM.

Conclusion: $\omega \otimes M^* \otimes M$ is CM

$$\begin{tikzcd}
\omega \otimes M^* \otimes M \arrow[r, "1 \otimes \theta"] \arrow[d] & \omega \otimes \mathrm{Hom}_R(M,M) \arrow[d] \\
\omega(M^* \otimes M) \arrow[r, "\simeq"] \arrow[dr, "\pi"'] & \omega \mathrm{Hom}_R(M,M) \arrow[d, "tr"] \\
 & \omega
\end{tikzcd}$$

$a(f \otimes x) \longmapsto af(x)$

$Q \subseteq R$

π onto $\overset{NAK}{\Longrightarrow}$ $M^*(M) = R$

$\Rightarrow R \mid M$. //