JAN-ERIK ROOS (p.1) 02-04-03

$(R, \mathfrak{m})$ commutative RLR

M, N f.g. R-modules $\quad \ell(M \otimes_R N) < \infty$

$$\chi_R(M,N) = \sum_{i=0}^{\dim R} (-1)^i \ell(\mathrm{Tor}_i^R(M,N)) \geq 0 \qquad \dim M + \dim N \leq \dim R$$

$$R = \frac{k[[X,Y,Z,W]]}{(XY-ZW)} \qquad N = R/(X,Z)\,; \ \mathrm{pd}_R(M) < \infty \qquad \underset{\overset{\|}{0}}{\dim M} + \underset{\overset{\|}{2}}{\dim N} < 3$$

$\chi_R(M,N) = -1$

Gulliksen proved: R local complete intersection, M, N $\ell(M \otimes N) < \infty$

$$\sum_{i \geq 0} \ell(\mathrm{Tor}_i^R(M,N)) z^i = \frac{\mathrm{Pol}^{M,N}(z)}{(1-z^2)^{\alpha}}$$

String Theory

$S^1 \longrightarrow X \qquad \mathrm{Tor}_*^R(M,N)$

$H^*(\mathcal{L}X; \mathbb{Q})$

$$\begin{tikzcd}
\mathcal{L}X \arrow[rr, bend left] \arrow[d] & f \in \arrow[d] & X^I \arrow[d] \\
X \arrow[rr, bend right, "\Delta"'] & (f(0), f(1)) & X \times X
\end{tikzcd}$$

$I = [0,1]$

$f : I \longrightarrow X$

$\dim X = 4$

$\pi_1(X) = 0$

$\boxed{H^2(X,\mathbb{Q}) > 2}$

Hochschild Homology

$$E^2_{pq} = \mathrm{Tor}_p^{H^*(X) \otimes H^*(X)}(H^*(X), H^*(X)) \Rightarrow H^*(\mathcal{L}X)$$

$$\begin{tikzcd}
H^*(\mathcal{L}X, \mathbb{Q}) & H^*(X, \mathbb{Q}) \arrow[l] \\
H^*(X) \arrow[u] & H^*(X) \otimes H^*(X) \arrow[l] \arrow[u]
\end{tikzcd}$$

(J.E.R. p 2)

$\dim H^2(X) = 3$

$H^2(X) \otimes H^2(X) \longrightarrow H^4(X) \cong k$

$\dfrac{k[X_1, X_2, X_3]}{(X_1^2 - X_2^2, X_2^2 - X_3^2, X_i X_k)_{i \neq k}}$ $X_1^2 + X_2^2 + X_3^2$

Koszul algebra

Hochschild homology

Let Γ be an algebra $/k$

$\left.\begin{array}{l} \Gamma \otimes_k \Gamma^\circ \longrightarrow \Gamma \to 0 \\ HH_*(\Gamma) = \mathrm{Tor}_*^{\Gamma \otimes_k \Gamma^\circ}(\Gamma, \Gamma) \end{array}\right\}$ Connes

Connes C_* $\cdots \to \overset{②}{\Gamma \otimes \Gamma \otimes \Gamma} \xrightarrow{b} \overset{①}{\Gamma \otimes \Gamma} \to \overset{⓪}{\Gamma}$

$b(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_n) + (-1)^n (a_n a_0 a_1 \ldots a_{n-1})$

$\quad = (a_0, \ldots, a_n)$

$H_*\left(\dfrac{C_*}{(1-t)}\right) = HC_*(\Gamma)$ $HC_*(k) = k \oplus 0 \oplus k \oplus 0 \cdots$

$\overline{HC}_n(\Gamma)$

k char 0, R graded $k \oplus \cdots$

$0 \to \underbrace{\overline{HC}_{n-1}(R)}\to \underbrace{HH_n(R)}_{n \geq 2} \to \underbrace{\overline{HC}_n(R)} \to 0$

$\overline{HC}_n(T(V)) = 0 \quad (n > 0)$

Example $\mathrm{Ext}_\Gamma^*(k,k) = \Gamma^!$

$\Gamma = \dfrac{k[X,Y]}{(X^2, XY)}$ $\Gamma^\circ = \dfrac{k[Z,U]}{(ZU, U^2)}$ $R = \Gamma \otimes_k \Gamma^\circ = \dfrac{k[X,Y,Z,U]}{(X^2, XY, ZU, U^2)}$

$M = R/(X-U, Y-Z)$

$\Gamma^! = \dfrac{k\langle x,y\rangle}{(y^2)}$ $\mathrm{Tor}_{i,q}^R(M,M) = HH_{i,q}(\Gamma)$ $\mathrm{Tor}_{i,j}^\Gamma(k,k) = 0 \quad i \neq j$

$0 \to \overline{HC}_{n-1,q}(\Gamma) \to HH_{n,q}(\Gamma) \to \overline{HC}_{n,q}(\Gamma) \to 0$

(J.E.R. p3)

Feigin - Tsygan: if Γ is a Koszul algebra $/k$, then
$\overline{HC}_{p,q}(\Gamma) \cong (\overline{HC}_{q-p-1,q}(\Gamma^!))^*$ $\quad 0,q \longrightarrow q-1,q$

$p,p+1$
$p,p+2$

$$\Gamma^! = \frac{k\langle X,Y\rangle}{(Y^2)} \simeq \underset{\overset{\parallel}{C}}{k\langle X\rangle} \amalg \frac{k\langle Y\rangle}{(Y^2)} = D \qquad (C^+ \otimes D^+)(z) = \frac{z^2}{1-z}$$

$\frac{k[X]}{X^2}$

$$\overline{HC}_n(C \amalg D) = \overline{HC}_n(C) \oplus \overline{HC}_n(D), \quad n > 0$$
$$\overline{HC}_0(C \amalg D) = \overline{HC}_0(C) \oplus \overline{HC}_0(D) + \underbrace{\overline{HC}_0(T(C^+ \otimes D^+))}$$

$\frac{R^+}{[R,R]}$

$(R, d) \to A$

$\Gamma = T(V) \qquad V_1 \oplus V_2 \oplus \cdots \qquad V(z)$

$$\overline{HC}_{0,*}(\Gamma)(z) = -\sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \log(1 - V((-1)^{n-1} z^n))$$

irrational function

$$\varphi(n) = |(\mathbb{Z}/n\mathbb{Z})^*| \qquad \frac{V'(z)}{1-V(z)} = \sum_{i \geq 0} c_i z^i \qquad \begin{matrix} c_0 > 0 \\ c_i > c_0 \\ \text{even } i > 0 \end{matrix}$$

$\frac{1}{1-z}$

SKOLEM - Mahler - Lech Theorem $\qquad$ k char $k = 0$

$$\frac{P(z)}{Q(z)} = \sum_{i \geq 0} b_i z^i \Longrightarrow \{ i \mid b_i = 0 \}$$

rational

union of a finite set and a finite set of arithmetic sequences

Macaulay 2

$$\Gamma = \frac{k[X_1, \ldots, X_n]}{(f_1, \ldots, f_t)} \qquad \overline{HC}_{i,j}(\Gamma) \qquad \underbrace{\Gamma \otimes \Gamma^o}_{R} = \frac{k[X_1, \ldots, X_n, Y_1, \ldots, Y_n]}{(f_1, \ldots, f_t, \tilde{f}_1, \ldots, \tilde{f}_t)}$$

$$M = R/(X_i - Y_i) \quad 1 \leq i \leq n$$

$$\mathrm{Tor}^R_{i,*}(M,M) \cong \overline{HC}_{i-1,j}(\Gamma) \oplus HC_{i,j}(\Gamma)$$

(J.E.R. p.4)

Conjecture $\sum |HH_{i,j}(R)| x^i y^j$ is irrational

Question

$$\begin{tikzcd}
 & k\langle X_1, \ldots, X_n\rangle / X_1^2+\cdots+X_n^2 & \\
k\langle X_1, \ldots, X_{n-1}\rangle \arrow[ur] & & k\langle X_n\rangle \arrow[ul] \\
X_1^2+\cdots+X_{n-1}^2 & k[S]_2 \arrow[ul] \arrow[ur] & X_n^2 \\
 & S \arrow[ul, mapsto, bend left] \arrow[ur, mapsto] &
\end{tikzcd}$$