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Mara Neusel: Applications of Steenrod technology to invariant theory

§ Motivation:

G finite group

$\mathbb{F}$ field

$\rho : G \hookrightarrow GL(n, \mathbb{F})$ linear rep.

$V = \mathbb{F}^n$

G acts on $\mathbb{F}[x_1, \ldots, x_n] = \mathbb{F}[V]$ via ρ

ring of invariants

(ρ is implicit)

$$\mathbb{F}[V]^G \subseteq \mathbb{F}[V]$$

Consider finite fields

$\mathbb{F}$

char $\mathbb{F} = p$, $|\mathbb{F}| = q$

If $|\mathbb{F}| < \infty$, then $\mathbb{F}[V]^G$ is an unstable invariant over the Steenrod algebra.

huh?

§ Steenrod algebra $\mathcal{P}^*$

$\mathcal{P}^*$ = Steenrod algebra of reduced powers

$= \mathbb{F}$-algebra generated by $\mathbb{P}^i$, $i \in \mathbb{N}_0$,

mod so-called Adem-relations.

How does $\mathcal{P}^*$ act on $\mathbb{F}[V]$?

Define $P[\xi] : \mathbb{F}[V] \longrightarrow \mathbb{F}[V][[\xi]]$,

- algebra homomorphism
- $P[\xi](l) = l + l^q \xi$

→ Give ξ degree $1-q$, then $P[\xi]$ has degree 0.

$$
\begin{aligned}
P[\xi](x_1 \cdot x_2) &= P[\xi](x_1) \cdot P[\xi](x_2) \\
&= (x_1 + x_1^q \xi)(x_2 + x_2^q \xi) \\
&= \underbrace{x_1 x_2}_{P^0(x_1 x_2)} + \underbrace{(x_1^q x_2 + x_1 x_2^q)}_{P^1(x_1 x_2)} \xi + \underbrace{x_1^q x_2^q}_{P^2(x_1 x_2)} \xi^2 .
\end{aligned}
$$

$$P[\xi](f) = \sum_{i=0}^{\deg(f)} P^i(f)\, \xi^i$$

Separate out homog. coords

→ P^i's are $\mathbb{F}$-linear: reduced Steenrod powers of f.

OBS1) $P^i(f) = \begin{cases} f^q, & i = \deg f \\ 0, & i > \deg f \end{cases}$ – unstability condition

OBS 2) $$P^i(f \cdot g) = \sum_{\alpha + \beta = i} P^\alpha(f) P^\beta(g)$$ – Cartan formula

$\mathcal{P}^* =$ $\mathbb{F}$-subalgebra of the endomorphisms of the functor $\mathbb{F}[-]$ generated by the P^i's, $i \in \mathbb{N}_0$.

By definition: $\mathbb{F}[V]$ carries an unstable action of the Steenrod algebra satisfying Cartan formula.

in general: H^* commutative graded $\mathbb{F}$-algebra
If H^* admits a $\mathcal{P}^*$-action satisfying the Cartan formula, then H^* is an algebra over the Steenrod algebra.

If In addition the unstability condition holds for H^*, H^* is an unstable algebra over $\mathcal{P}^*$.

$\frac{f}{g} \in \mathbb{F}[V]$

$$P[\xi]\left(\frac{f}{g}\right) = \frac{P[\xi](f)}{P[\xi](g)}$$

Just an algebra over $\mathcal{P}^*$.

$\mathbb{F}[V]^G$ inherits (unstable) $\mathcal{P}^*$-action from $\mathbb{F}[V]$

§ Primary decomposition

H^* unstable Noeth. algebra / $\mathcal{P}^*$

$I \subseteq H^*$ $\mathcal{P}^*$-invariant ideal (i.e. closed under $\mathcal{P}^*$-action)

Thm 1: $\exists$ a primary decomposition $I = q_1 \cap \cdots \cap q_e$ of I, such that q_i's, $\sqrt{q_i}$'s are $\mathcal{P}^*$-invariant.

want to look at H^*-modules over the Steenrod algebra: $\mathcal{U}(H^*)$

H^* unst. Noeth. algebra / $\mathcal{P}^*$
H^*-modules M with an unstable action of $\mathcal{P}^*$.

$$P^i(h \cdot m) = \sum_{\alpha+\beta=i} P^\alpha(h) \cdot P^\beta(m) \quad \text{(Cartan)}$$

and $P^i(m) = 0$ for $i > \deg(m)$

Thm 2: $M' \subseteq M$ fin. gen. H^*-modules / $\mathcal{P}^*$
$\Rightarrow \exists$ primary decomp $M' = Q_1 \cap \cdots \cap Q_e$,
where Q_i's, $\sqrt{Q_i}$'s are in $\mathcal{U}(H^*)$.

Classically,
$\mathfrak{p}$ assoc. prime $= (M' : t)$, $t \in M$

$$H/\mathfrak{p} \hookrightarrow M/M', \quad 1 \mapsto t$$

(classically $H/\mathfrak{p} \cong Ht \subseteq M/M'$) Problem: Why is Ht closed under the $\mathcal{P}^*$-action?

Thm 3: $\exists t,\ p = (M' : t)$ s.t. $Ht \in \mathcal{U}(H^*)$

Corollary: (prime filtration)
H^* a Noeth. unstable alg, M f.g., $M \in \mathcal{U}(H^*)$, then
M has a prime filtration

i.e. M is an $H^* \otimes \wp^*$ module

$$0 = M_0 \hookrightarrow M_1 \hookrightarrow \cdots \hookrightarrow M_\ell = M, \text{ where}$$

$$M_i / M_{i-1} = \Sigma^{(\deg t)} H/p_i \cong Ht_i \in \mathcal{U}(H^*)$$

($H^* \to H^* \otimes \wp^* \leftarrow \wp^*$ are not necessarily maps in the category)

§ Invariant theory again

Landweber–Stong conjecture:

Dickson 1911 $\mathbb{F}[V]^{GL(n,\mathbb{F})} = \mathbb{F}[d_{n,0}, \ldots, d_{n,n-1}]$, $\deg d_{n,i} = q^n - q^i$

$$\begin{array}{c} \mathbb{F}[V]^{GL(n,\mathbb{F})} \\ \downarrow \\ \mathbb{F}[V]^G \\ \downarrow \\ \mathbb{F}[V] \end{array}$$

depth $\mathbb{F}[V]^G \geq k \iff d_{n,n-1}, \ldots, d_{n,n-k}$ is a regular sequence on $\mathbb{F}[V]^G$

Now a theorem: D Bour -, S. Zarati 1997 (Inventiones)

- $\mathbb{F}[V]^G$ is in $\mathcal{U}(\mathbb{F}[V]^{GL(n,\mathbb{F})})$
- $\mathbb{F}[V] \otimes_{\mathbb{F}[V]^{GL_n \mathbb{F}}}$ — $\rightsquigarrow$ object M in $\mathcal{U}(\mathbb{F}[V])$
- Translate into a statement about $E(M)$ injective hull.
- Injectives are classified; prove statement here (induction on length of V)
- press this down to M.

instead:

$\mathbb{F}[V]^G$ is in $\mathcal{U}(\mathbb{F}[V]^{GL_n \mathbb{F}})$

$\rightsquigarrow$ prime filtration

use double induction on depth + length of filtration

Landweber: $(0) \subseteq (d_{n,0}) \subseteq (d_{n,0}, d_{n,1}) \subseteq \cdots \subseteq (d_{n,0}, \ldots, d_{n,n-1}) \subseteq \mathbb{F}[V]^{GL}$

these are all of the $\mathcal{P}^*$-invariant prime ideals in the Dickson algebra.

Landweber: $\mathbb{F}[V]^{GL} \hookrightarrow \mathbb{F}[V]^G \hookrightarrow \mathbb{F}[V]$ (Neusel) Serre: ~~$\mathcal{P}^*$-invariant prime~~ $\mathfrak{p}$ = (linear forms)

$(d_{n,0}, \ldots, d_{n,i}) \longleftarrow \mathfrak{p}' \ \mathcal{P}^*$-invar $\longleftarrow$