Entanglement properties of Gaussian states

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MPQ

Gaussian states

A

$\dim(H) = \infty$

$[X_A, P_A] = i$

ρ is Gaussian if it can be written as:

$$\rho = k e^{-Q(X_A, P_A)}$$

$Q \geq 0$ is a polynomial of degree 2

A B C

$\dim(H_n) = \infty$

$[X_n, P_n] = i$

$H = \otimes_n H_n$

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$Q \geq 0$ is a polynomial of degree 2

Some interesting Gaussian states:

1. Coherent and squeezed states: $|\alpha\rangle \propto \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$

2. Two-mode squeezed states: $|\Phi_r\rangle \propto \sum_{n=0}^{\infty} \tanh^n(r) |n\rangle_A \otimes |n\rangle_B$

$$\frac{1}{2}\left[\Delta\left(X_A - X_B\right) + \Delta\left(P_A + P_B\right)\right] < 1$$

3. Thermal states: $\rho \propto \sum_{n=0}^{\infty} e^{-\kappa n} |n\rangle\langle n|$

Why are Gaussian states interesting in Quantum Information?

1. They are relatively simple to handle mathematically.

Density operator	ρ	→	γ	(correlation matrix)
Hilbert space	H	→	X	(symplectic vector space)

2. Display most of the interesting phenomena in QI.

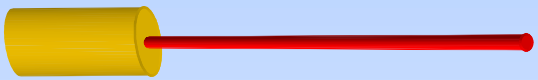
Entanglement (bound and free)
Teleportation, Quantum Cryptography

3. Can be easily prepared and manipulated in experiments.

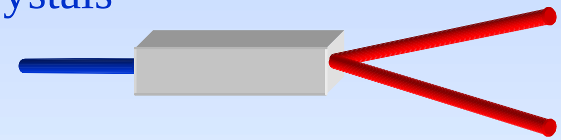
1. Optics

Sources:

Lasers

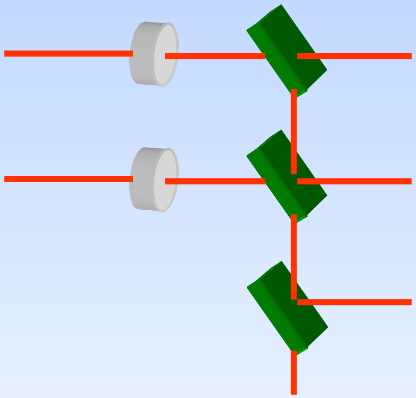


Crystals



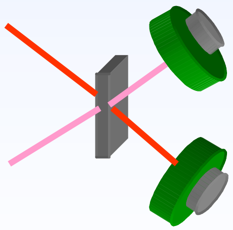
Passive elements:

Fibers, lenses, beam splitters,
polarizers, lambda-plates, etc



Measurements:

Homodyne detectors

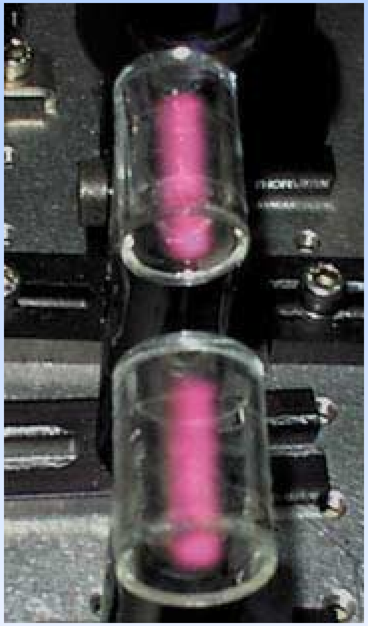


Decoherence:

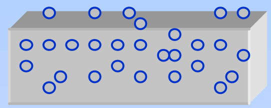
Photon absorption, phase-shifts

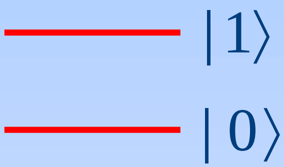
Gaussian states can be prepared, manipulated, and measured

2. Atomic ensembles:



Two atomic ensembles
(Polzik et al, Arhus)





$$\left(\mathbb{C}^{2}\right)^{\otimes N} \rightarrow H$$

Operators

$$J_{y,z} = \frac{1}{2}\sum_{k=1}^{N} \sigma_{y,z}^{(k)} \qquad X = \sqrt{\frac{2}{N}} J_y \quad P = \sqrt{\frac{2}{N}} J_z$$

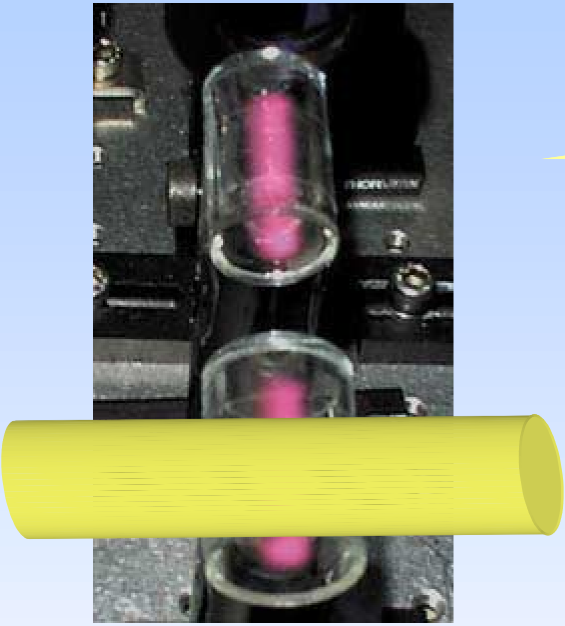
$$[J_y, J_z] = iJ_x \rightarrow [X,P] = i\frac{2}{N} J_x \simeq i$$

$$\langle J_x \rangle \simeq N/2$$

States

$$\left(|0\rangle + |1\rangle\right)^{\otimes N} \rightarrow \text{Coherent state } |0\rangle$$

2. Atomic ensembles:



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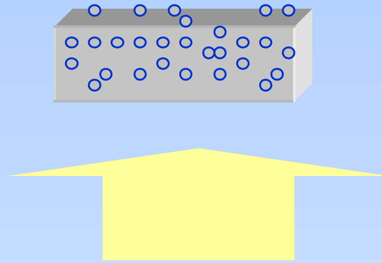
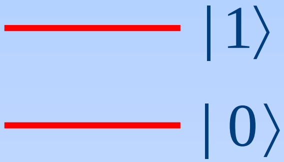




The light can analogously be described

$$H = gX_A X_L$$

Both, light and atoms, can be manipulated independently according to

$$H_{\text{Local}} = aX + bP + c(X^2 + P^2)$$

Using magnetic fields/polarizators and plates

Outline

Separability:
General case
(with M Lewenstein)

Separability:
Two modes
(with L.M. Duan and P. Zoller)

Gaussian
Transformations

Distillability
with Gaussian op.

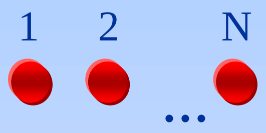
Distillability
(with L.M. Duan and P. Zoller)

Entanglement
generation

Pure state
transformations
(with J. Eisert and M. Plenio)

Entanglement
measures
(with M. Wolf and R. Werner)

1. Description



N-modes

$$[X_n, P_n] = i$$

$$H = \otimes_n H_n$$

$$\rho = k e^{-Q(X_n, P_n)}$$

All the information about the states is contained in the first and second moments:

$$\langle X_1 \rangle, \langle P_1^2 \rangle, \langle X_1 P_3 + P_3 X_1 \rangle \ldots$$

It is convenient to characterize Gaussian states by: where

$$R = (X_1, P_1, \ldots, X_N, P_N)$$

- Displacement vector: $d_\alpha = \langle R_\alpha \rangle$

- Correlation matrix: $\gamma_{\alpha,\beta} = \langle (R_\alpha - d_\alpha)(R_\beta - d_\beta) \rangle + c.c.$

One mode, d=0:

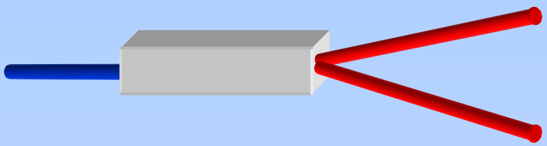
$$\gamma = \begin{pmatrix} 2\langle X^2 \rangle & \langle XP + PX \rangle \\ \langle XP + PX \rangle & 2\langle P^2 \rangle \end{pmatrix}$$

Two modes or two systems

$$\gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}$$

Example: Two-mode squeezed states.

$$\rho = |\Phi_r\rangle\langle\Phi_r|$$



$$|\Phi_r\rangle \propto \sum_{n=0}^{\infty} \tanh^n(r)\,|n\rangle \otimes |n\rangle$$

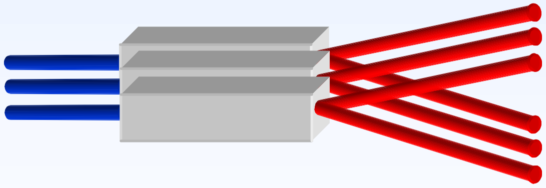
Correlation matrix:

$$\gamma = \begin{pmatrix} A_r & C_r \\ C_r^T & B_r \end{pmatrix}$$ where $$A_r = B_r = \cosh(r)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C_r = \sinh(r)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note that for $r \to \infty$ we have an (improper) maximally entangled state, i.e., maximally entangled states are (limit) of Gaussian states.

Example: NxN-mode squeezed states.



$$\gamma = \begin{pmatrix} A_r & C_r \\ C_r^T & B_r \end{pmatrix}$$ where $$A_r = B_r = \cosh(r)\,\mathrm{P}$$

$$C_r = \sinh(r)\Lambda$$

$$\Lambda = \sigma_z \oplus \sigma_z \oplus \ldots$$

- Given $\rho \propto e^{-H(X_1,P_1,X_2,P_2,\ldots)}$ it is very simple to determine the displacement and the correlation matrix.

- Given the displacement vector and the correlation matrix, one can also determine ρ

$$\rho = \pi^{-N} \int_{\mathbb{R}^{2N}} dx\, e^{-\frac{1}{4}x^T \gamma x + i d^T x} W(x)$$

where

$W(x) = e^{-ix^T R}$ are the Weyl operators.

When is γ a correlation matrix?

For valid density operators: $\gamma = \gamma^T \geq iJ$

where $J = J_2 \oplus J_2 \oplus \ldots$ is the „symplectic matrix“

and $J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

2. Entanglement

1 2 … N — N-modes

$\rho_0 = k e^{-Q(X_n, P_n)}$

1 2 … M — M-modes

$$\gamma_0 = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}$$

(A: 2nX2n; B: 2mX2m)

Given a CM, γ_0: does it correspond to a separable state (separable)?

What is known?

For N=M=1, partial transposition is a necessary and sufficient condition.

(L.M. Duan, G. Giedke, I.Cirac and P. Zoller, 2000, Simon, 2000)

For N=M=2, there exist PPT entangled states

(R. Werner and M. Wolf,2001))

Separability criterion

(G. Giedke, B. Kraus, M. Lewenstein, and Cirac, 2001)

- Idea: define a map

$$\gamma_0 = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}$$

2nX2n (block A), 2mX2m (block B)

$$\gamma_0 \rightarrow \gamma_1 \rightarrow \gamma_2 \rightarrow \dots \rightarrow \gamma_N$$

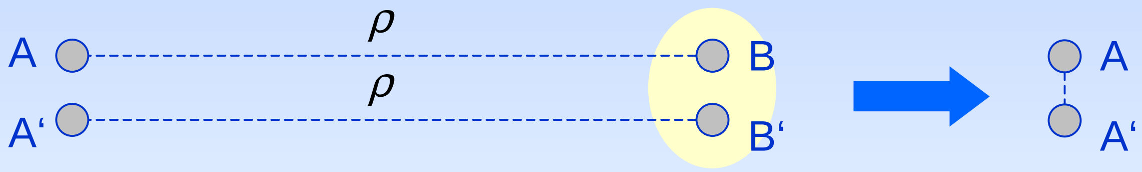
$$A_{N+1} := B_{N+1} := A_N - \Re\left[C_N (B_N - iJ)^{-1} C_N^T \right]$$

$$C_{N+1} := -\Im\left[C_N (B_N - iJ)^{-1} C_N^T \right]$$

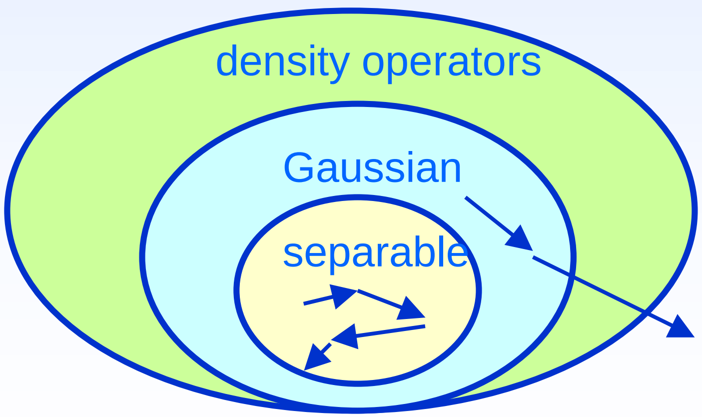
- Facts:
  - γ_N is a CM of a separable state iff γ_{N+1} is too.
  - If γ_N is a CM of an entangled state, then either
    - γ_{N+1} is no CM
    - or
    - γ_{N+1} is a CM of an entangled state
  - If γ_0 is separable, then $\gamma_N \rightarrow \gamma_\infty$. This last corresponds to $\rho_\infty = \rho_A \otimes \rho_B$ (for which one can readily see that is separable)

Connection to positive maps?

- Map for CM's: $\gamma_N \to \gamma_{N+1}$
- Map for density operators: $\rho_N = e^{-H_N} \to \rho_{N+1} = e^{-H_{N+1}}$



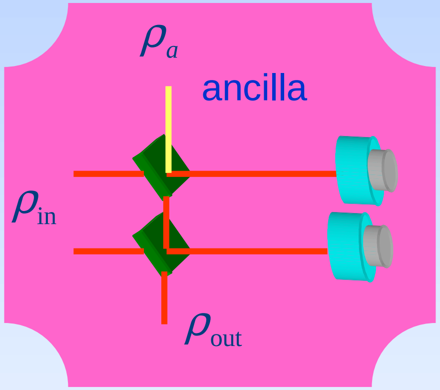
Is non-linear!

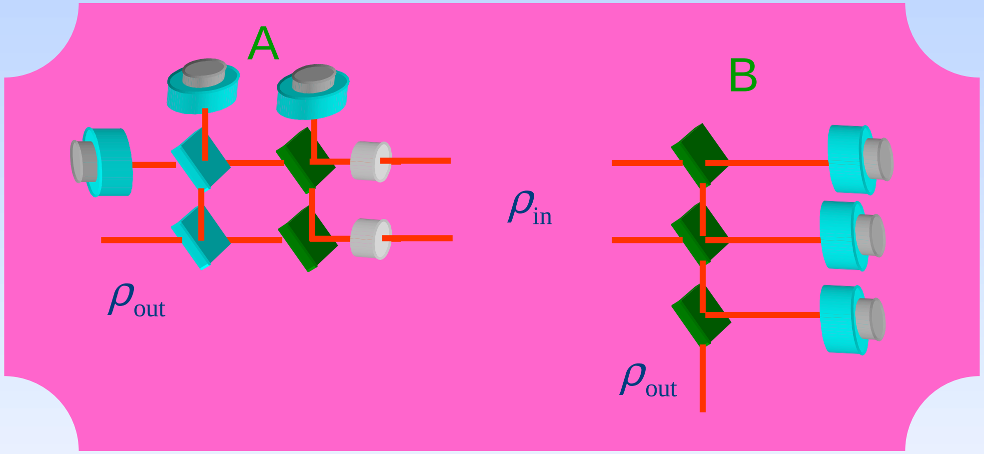


3. Gaussian transformations

What can we do with these systems?

Mathematical description of physical operations: $\rho_{\text{in}} \rightarrow \rho_{\text{out}}$





It is difficult:

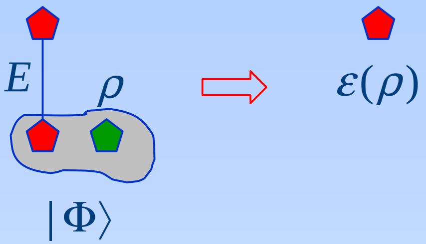
Mathematical description excluding measurements: Demoen et al, 1977

We want to know:

- -Which operations transform Gaussian states into Gaussian states.
- -If all of them can be performed with the tools available in the lab.
- -Which can be performed locally (GLOCC).

Characterization of Gaussian operations ε

Physical explanation

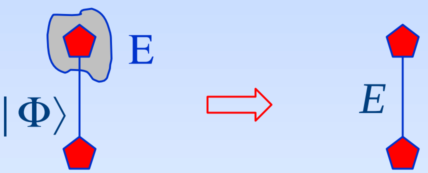


Idea: Use the identity (Jamiolkovski)

$$\varepsilon(\rho) = \mathrm{tr}_{2,3}[(E_{1,2} \otimes \rho_3)(|\Phi\rangle_{2,3}\langle\Phi|)]$$

where

$$E = (\varepsilon \otimes 1)(|\Phi\rangle\langle\Phi|)$$



For Gaussians:

$|\Phi\rangle$ are Gaussian states

ε is a Gaussian operation

$\Rightarrow E$ is a Gaussian state

$\Rightarrow E$ whatever it is, it can be characterized by Γ, D

$$\gamma' = \tilde{\Gamma}_1 - \tilde{\Gamma}_{12} \frac{1}{\tilde{\Gamma}_2 + \gamma} \tilde{\Gamma}_{12}^T$$

Remarks:

$$\gamma' = \tilde{\Gamma}_1 - \tilde{\Gamma}_{12}\frac{1}{\tilde{\Gamma}_2+\gamma}\tilde{\Gamma}_{12}^T$$

- γ' is a non-linear function of γ

- All Gaussian operations can be implemented in the lab, since E can be prepared in the lab, and Bell measurements can be performed

- For two or more systems, the operation is a GLOCC iff Γ is separable:

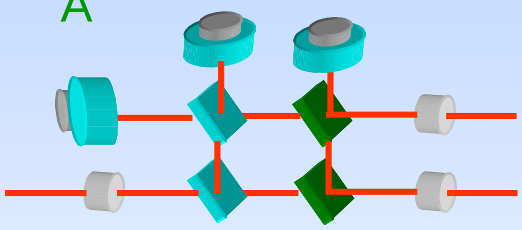
Cirac, Dür, Kraus, and Lewenstein, 2001

4. Gaussian distillation

Gaussian distillation:

Can we distill using the tools that are available in the lab?
(beam splitters, polarizers, homodyne measurements, etc)

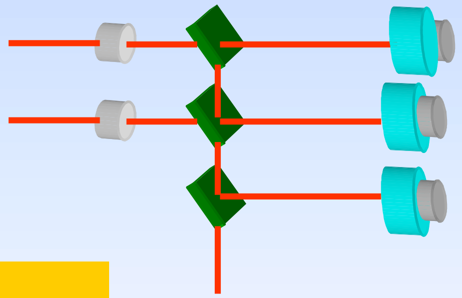
A



ρ

Gaussian:

B



$$\rho_{\text{in}} = \rho^{\otimes N} \rightarrow \rho_{\text{out}} = |\Psi\rangle\langle\Psi|$$

Gaussian distillation is a relevant open problem since it is required for long distance quantum communication using quantum repeaters

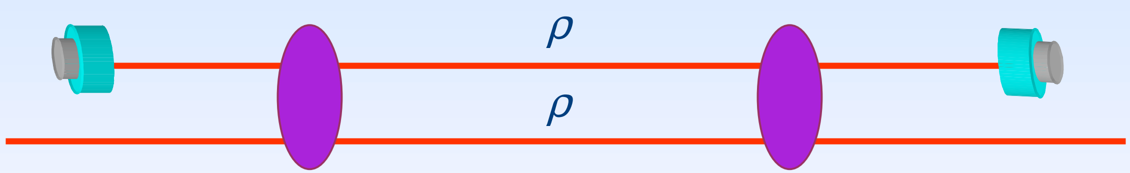
Note that

- In general, the distillability problem has been solved!

Gaussian states are distillable iff they are NPT: (Giedke,Kraus,Lewenstein and Cirac. 2001)

But we are considering here only Gaussian operations.

- Eisert, Sheel, and Plenio have shown that the negativity of two symmetric copies in 1x1 modes cannot increase using some particular operations: Eisert, et al 2002



(set-up used for two qubits: Bennet et al, 93)

But:

- Distillation could be possible with non-symmetric states.
- With more than two copies.
- Using other operations.
- Other measures of entanglement may increase.

We show that $\rho^{\otimes N} \to |\Phi\rangle\langle\Phi|$ is impossible with GLOCC

Idea:

1) We define a quantity, $p(\gamma)$ related to the entanglement of the state:

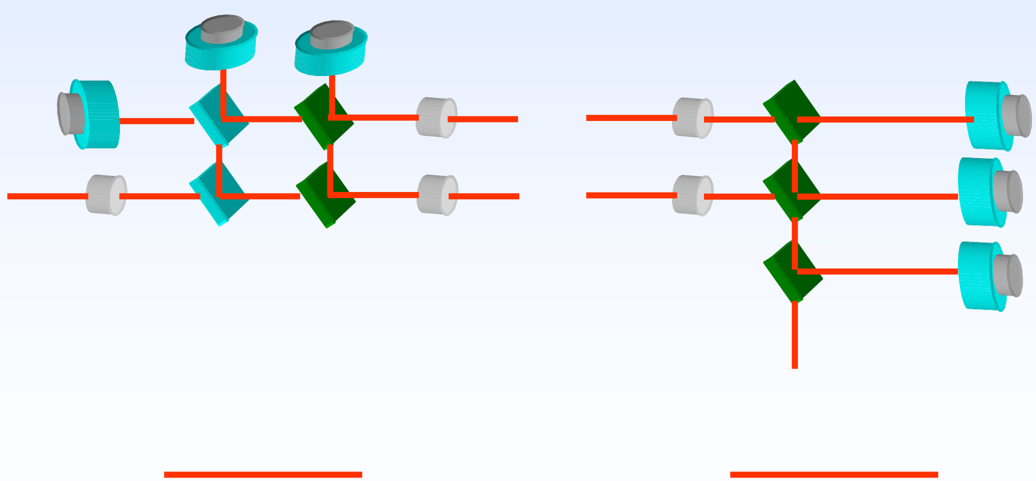
$p(\gamma) = 1$ if γ is separable.

$p(\gamma) < 1$ if γ is inseparable.

$p(\gamma) = 0$ if γ is maximally entangled.

$p(\gamma \oplus \gamma \oplus \ldots) = p(\gamma)$

2) $p(\gamma)$ cannot be decreased by GLOCC



ρ $\quad p(\gamma) = p_0 > 0$

$\rho^{\otimes N}$ $\quad p(\gamma \oplus \gamma \oplus \ldots) = p_0 > 0$

GLOCC (crossed out)

$|\Phi\rangle$ $\quad p(\gamma) \to 0$

Gaussian measure of entanglement: $p(\gamma)$

- For separable states we know that there exist $\gamma_{A,B} \geq iJ$ such that

$$\gamma \geq \gamma_A \oplus \gamma_B$$

(R. Werner and M. Wolf, 00)

- For entangled states we can always find $\gamma_{A,B} \geq iJ$ and $p \in [0,1)$ such that

$$\gamma \geq p(\gamma_A \oplus \gamma_B)$$

(for example, take p=0).

- We take the maximum *p* (smaller or equal to 1), and call it $p(\gamma)$

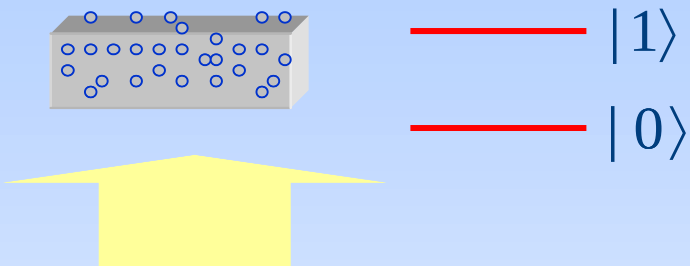
-For separable states: $p(\gamma) = 1$
-For entangled states: $p(\gamma) < 1$

Properties:

- Cannot increase by GLOCC
- Includes PPTES.
- Can be computed.

5. Entanglement generation

(Kraus, Hammerer, Giedke, Cirac, quant-ph2002)



We have at our disposal:

Interaction: $H_0 = gX_A X_L$

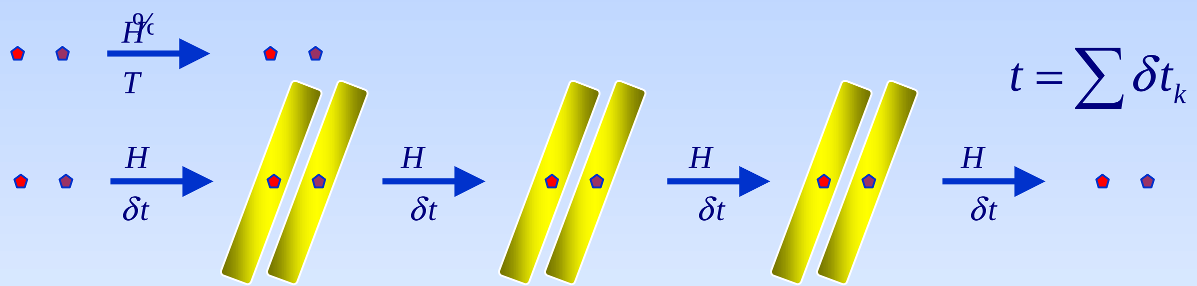
Certain local operations: $H_{\mathrm{Local}} = aX + bP + c(X^2 + P^2)$

What can we do?

- Which operations and states can be generated?
- How to entangle these systems optimally?

5.1 Hamiltonian simulation:

Given $H = (X_A, P_A) K (X_B, P_B)$ and H_{Local} we want to study under which conditions one can simulate $\tilde{H} = (X_A, P_A)\tilde{K}(X_B, P_B)$



(for qubits, see Bennet et al, 2002)

Writting $K = O \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \tilde{O}$ where $s_1 \geq |s_2|$ and $O, \tilde{O} \in so(2)$

we have the necessary and sufficient conditions:

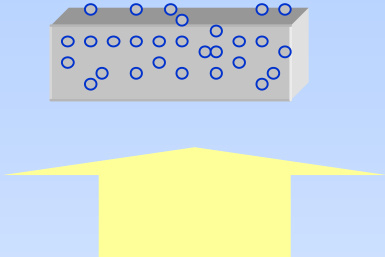
$$(s_1 + s_2)t \geq (\tilde{s}_1 + \tilde{s}_2)T$$
$$(s_1 - s_2)t \geq (\tilde{s}_1 - \tilde{s}_2)T$$

- Every H can simulate H‘ except if $s_1 = \pm s_2$
- The original interaction is universal.
- Any Gaussian operation can be implemented.
- Any Gaussian state can be created.

Application: quantum memory: $H_0 = X_A P_L - P_A X_L$

Application: spin squeezing: $|\Psi\rangle_A = |\Phi_r\rangle$

5.2 Optimal generation of entanglement



Given $H = (X_A, P_A) K (X_B, P_B)$, H_{Local} and some initial state γ, we want to generate entanglement.

Entanglement rate: $\Gamma_E := \lim_{\delta t \to 0} \left. \frac{E[\gamma(\delta t)] - E(\gamma)}{\delta t} \right|_{\text{opt}} \propto s_1 e^{l} - s_2 e^{-l}$

(for qubits, see Dür,Vidal,Cirac,Linden and Popescu, 2001)

where $\cosh(2l) = \frac{\det(A)}{-2\det(C)} \operatorname{tr}\left(A^{-2} C C^T\right)$ $\qquad \gamma_0 = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}$

- Some Hamiltonians cannot produce entanglement if there is no initial squeezing $l = 0$
- Entanglement is more efficiently created if the systems are squeezed $l \neq 0$
- The systems have to be „properly squeezed“.
- The rate is not bounded.
- For the an unsqueezed initial state and the physical Hamiltonian, $\left. E(t) \right|_{\text{opt}} = t$