Hidden Translation and Orbit Coset in Quantum Computing

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joint work with

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Structure of the talk

- HIDDEN TRANSLATION

  Efficient quantum algorithm in elementary Abelian groups

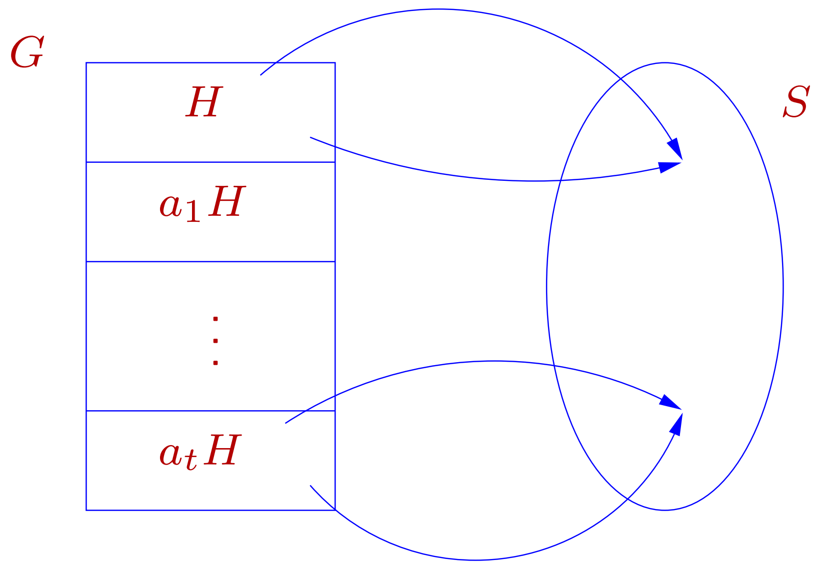
- ORBIT COSET

  Efficient recursive quantum algorithm in smoothly solvable groups

Input: G finite group, and $f : G \to S$ hiding $H \leq G$:

$\forall x \in G, h \in H, f(x) = f(xh)$ and $\forall x, y \in G,\ xH \neq yH \implies f(x) \neq f(y)$.

Output: Generators for H.



Theorem: Can be solved in quantum $\mathrm{poly}(\log|G|)$-time when

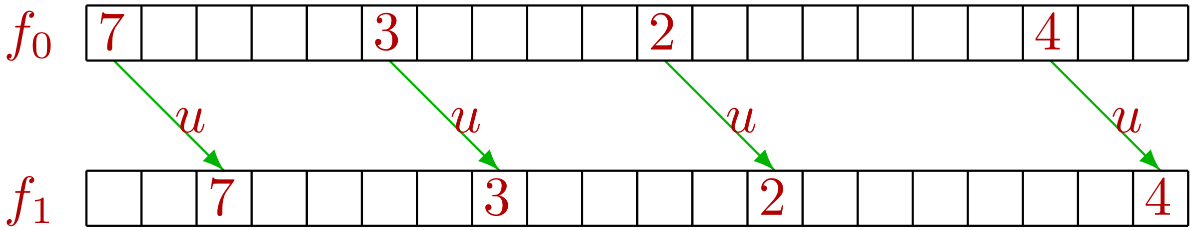
- $G = \mathbb{Z}_2^k \wr \mathbb{Z}_2$ [Roetteler,Beth'98]
- H is normal and QFT_G is available [Hallgren,Russell,Ta-Shma'00]
- H is normal and G is solvable [Ivanyos,Magniez,Santha'01]
- $\cap\{N(H) \ : H \leq G\}$ is large [Grigni,Schulman,Vazirani,Vazirani'01]
- $G = \mathbb{Z}_p \rtimes \mathbb{Z}_q$ when $q = \frac{p-1}{(\log p)^c}$ [Moore,Rockmore,Russell,Schulman'02]
- $G = \mathbb{Z}_n \rtimes \mathbb{Z}_2$ with exponential postprocessing [Ettinger,Høyer'00]
- $G = \mathbb{Z}_p^n \rtimes \mathbb{Z}_2$ for fixed prime p

Input: G finite group.

$f_0, f_1 : G \to S$ injective functions having a translation $u \in G$:

$$\forall x \in G, \quad f_0(x) = f_1(xu).$$

Output: u.



Theorem. [Ettinger-Høyer'00]. If G finite Abelian group then HIDDEN TRANSLATION on $G \simeq$ HIDDEN SUBGROUP on $G \rtimes \mathbb{Z}_2$.

Group operation on $G \rtimes \mathbb{Z}_2$: $(x_1, b_1) \cdot (x_2, b_2) = (x_1 + (-1)^{b_1} x_2, b_1 \oplus b_2)$.
Fact. $f(x,b) = f_b(x)$ hides $H = \{(0,0); (u,1)\}$ on $G \rtimes \mathbb{Z}_2$.

Theorem. For every prime p, HIDDEN TRANSLATION can be solved on $\mathbb{Z}_p^n$ by a quantum algorithm with query complexity $O(p(n+p)^{p-1})$ and time complexity $(n+p)^{O(p)}$.

Idea of [EH'00]: Apply QFT on the direct product $\mathbb{Z}_p^n \times \mathbb{Z}_2$.

State: $$\frac{1}{2p^n} \sum_{x \in \mathbb{Z}_p^n} \sum_{b=0}^{1} \sum_{y \in \mathbb{Z}_p^n} \sum_{c=0}^{1} \omega_p^{x \cdot y} (-1)^{bc} |y\rangle |c\rangle |f_b(x)\rangle$$

Rewrite using the hidden translation:

$$\frac{1}{2p^n} \sum_{x \in \mathbb{Z}_p^n} \sum_{y \in \mathbb{Z}_p^n} \sum_{c=0}^{1} \left(\omega_p^{x \cdot y} + \omega_p^{(x+u) \cdot y} (-1)^c\right) |y\rangle |c\rangle |f_0(x)\rangle$$

For all x, y the amplitude of $|y\rangle|1\rangle|f_0(x)\rangle$ is:

$$\frac{1}{2p^n} \omega_p^{x \cdot y} (1 - \omega_p^{y \cdot u})$$

After observation:

$$\Pr[\text{output} = (y, 1)] = \frac{1}{4p^{2n}} |1 - \omega_p^{y \cdot u}|^2.$$

Properties of the output distribution:

- $\Pr[c = 1] = \frac{1}{2}$
- depends only on $y \cdot u$
- for every $(y, 1)$ observed: $\quad y \cdot u \neq 0 \mod p$.

The algorithm: Part 2 (classical postprocessing)

Sample $(y, 1)$ such that $y \cdot u \neq 0 \mod p$ (*i.e.* $y \notin u^\perp$)

Linear inequations $\mapsto$ polynomial equations

$$y \cdot u \neq 0 \mod p \iff (y \cdot u)^{p-1} = 1 \mod p$$

Fact. Solving polynomial equations is NP-complete.

Idea: 'Linearize' the system in the symmetric power of $\mathbb{Z}_p^n$

Definition. $\mathbb{Z}_p^{(p-1)}[x_1, \ldots, x_n]$ is the vector space of homogeneous polynomials in n-variables of degree $(p-1)$ over $\mathbb{Z}_p$.

- A basis: Monomials of degree $(p-1)$
- Dimension: $\binom{n+p-2}{p-1}$

Transfer from $\mathbb{Z}_p^n$ via $(\mathbb{Z}_p^n)^*$ to $\mathbb{Z}_p^{(p-1)}[x_1, \ldots, x_n]$:

Definition. For $y = (a_1, \ldots, a_n) \in \mathbb{Z}_p^n$ let $y^{(p-1)} = (\sum_j a_j x_j)^{p-1}$.

$$y \cdot u \neq 0 \mod p \implies y^{(p-1)} \cdot u^* = (y \cdot u)^{p-1} = 1 \mod p,$$

where in $u^* \in \mathbb{Z}_p^n$ the monomial $x_1^{e_1} \cdots x_n^{e_n}$ has coordinate $u_1^{e_1} \cdots u_n^{e_n}$.

End of the algorithm:

- Hopefully the linear system in $\mathbb{Z}_p^{(p-1)}[x_1, \ldots, x_n]$ has unique

solution

- Find the solution $U = u^*$
- Try the $(p-1)$ candidates v such that $v^* = u^*$

Example. $p = 3$, $n = 3$, $u = (1, 2, 0)$.

Sample in $\mathbb{Z}_3^3$	Inequation in $\mathbb{Z}_3^3$	Equation in $\mathbb{Z}_3^{(2)}[x_1, x_2, x_3]$
$y_1 = (0, 1, 0)$	$x_2 \cdot u \neq 0$	$x_2^2 \cdot U = 1$
$y_2 = (0, 2, 1)$	$(2x_2 + x_3) \cdot u \neq 0$	$(x_2^2 + x_3^2 + x_2 x_3) \cdot U = 1$
$y_3 = (0, 2, 2)$	$(2x_2 + 2x_3) \cdot u \neq 0$	$(x_2^2 + x_3^2 + 2x_2 x_3) \cdot U = 1$
$\vdots$	$\vdots$	$\vdots$

where $x_1 = (1, 0, 0)$, $x_2 = (0, 1, 0)$, $x_3 = (0, 0, 1)$,
$x_1^2 = (1, 0, 0, 0, 0, 0)$, $\ldots$

System of full rank $\Longrightarrow$ unique solution $U = x_1^2 + x_2^2 + 2x_1 x_2$.

Try the 2 possible translations $(1, 2, 0)$ and $(2, 1, 0)$ $\rightsquigarrow$ $u = (1, 2, 0)$.

Translation finding$^f(\mathbb{Z}_p^n)$

0. If $f_0(0) = f_1(0)$ then return 0.
1. $N \leftarrow 13p\binom{n+p-2}{p-1}$.

2. For $i = 1, \ldots, N$ do $(z_i, b_i) \leftarrow$ **Fourier sampling**$^f(\mathbb{Z}_p^n \times \mathbb{Z}_2)$.
3. $\{y_1, \ldots, y_m\} \leftarrow \{z_i : b_i = 1\}$.

4. For $i = 1, \ldots, m$ do $Y_i \leftarrow y_i^{(p-1)}$.
5. Solve $Y_1 \cdot U = 1, \ldots, Y_m \cdot U = 1$.
6. If several solutions then abort.

7. Let j be such that the coefficient of x_j^{p-1} in U is 1.
8. Let $v \in \mathbb{Z}_p^n$ be such that $v_k v_j$ is the coefficient of $x_k x_j^{p-2}$ in U.

9. Find $0 < a < p$ such that $f_0(0) = f_1(av)$.
10. Return av.

Line Lemma. Let $L_{z,y} = \{(z+ay)^{(p-1)} : 0 \leq a \leq p-1\}$ for $y, z \in \mathbb{Z}_p^n$. Then $y^{(p-1)} \in \mathrm{Span}(L_{z,y})$.

Proof. Let $M_{z,y} = \{\binom{p-1}{k} z^{(k)} y^{(p-1-k)} : 0 \leq k \leq p-1\}$.
Claim: $\mathrm{Span}(L_{z,y}) = \mathrm{Span}(M_{z,y})$.

	$z^{(p-1)}$	$(z+y)^{(p-1)}$	$(z+2y)^{(p-1)}$	$\dots$	$(z+(p-1)y)^{(p-1)}$
$\binom{p-1}{0} z^{(p-1)}$	1	1	1	$\dots$	1
$\binom{p-1}{1} z^{(p-2)} y^{(1)}$	0	1	2	$\dots$	$(p-1)$
$\binom{p-1}{2} z^{(p-3)} y^{(2)}$	0	1	2^2	$\dots$	$(p-1)^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\binom{p-1}{p-1} y^{(p-1)}$	0	1	$(p-1)^2$	$\dots$	$(p-1)^{(p-1)}$

Corollary. $\mathbb{Z}_p^{(p-1)}[x_1, \dots, x_n]$ is spanned by $\{y^{(p-1)} : y \in \mathbb{Z}_p^n\}$.

Lemma. Let $W \leq \mathbb{Z}_p^{(p-1)}[x_1,\ldots,x_n]$ and $R = \{y \in \mathbb{Z}_p^n : y^{(p-1)} \in W\}$.
Set $V_k = \{y \in \mathbb{Z}_p^n : y \cdot u = k\}$, and $R_k = R \cap V_k$.
If $W \neq \mathbb{Z}_p^{(p-1)}[x_1,\ldots,x_n]$ then $\frac{|R_k|}{|V_k|} \leq \frac{p-1}{p}$ for $k = 1,\ldots,p-1$.

Proof. Corollary $\implies R \neq \mathbb{Z}_p^n$.

Case 1: $R_0 = V_0$. Then $R_k \neq V_k$ for $k = 1,\ldots,p-1$. Let $y \in V_1 - R_1$.
Line Lemma $\implies$ in each coset of $<y>$ an element is outside R.

	$<y>$	$\ldots$	$z + <y>$	$\ldots$
V_0	0	$\ldots$	z	$\ldots$
V_1	y	$\ldots$	$z + y$	$\ldots$
$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\ldots$
V_{p-1}	$(p-1)y$	$\ldots$	$z + (p-1)y$	$\ldots$

$\implies \frac{|R|}{|Z_p^n|} \leq \frac{p-2}{p-1} \implies \frac{|R_k|}{|V_k|} \leq \frac{p-2}{p-1}$.

Case 2: $R_0 \neq V_0$. Let $y \in V_0 - R_0$, then V_k is union of cosets of $<y>$.
Line Lemma $\implies \frac{|R_k|}{|V_k|} \leq \frac{p-1}{p}$.

G a finite group given by generators

Elements are encoded in $\{0,1\}^n$ where $n = O(\log |G|)$

Group operations are performed by oracles

For G solvable, the derived series is computable in probabilistic polynomial time [Babai et al.'95]

$$G = G^{(0)} \rhd G^{(1)} \rhd \ldots \rhd G^{(m)} = \{1_G\}$$

For G solvable, the composition series is computable in quantum polynomial time [Watrous'01] [Ivanyos et al.'01]

$$G = G_0 \rhd G_1 \rhd \ldots \rhd G_m = \{1_G\}$$

where $|G_i/G_{i+1}|$ is prime

G finite group, Γ mutually orthogonal quantum states

Action of G on Γ is a homomorphism

$$\begin{array}{rcl} \alpha \ : \ G & \rightarrow & \mathrm{Perm}(\Gamma) \\ x & \mapsto & \alpha_x \end{array}$$

Notation. $\alpha_x(|\varphi\rangle) = |x \cdot \varphi\rangle$

Oracle for a group action : $|x\rangle|\varphi\rangle \mapsto |x\rangle|x \cdot \varphi\rangle$

Example 1. For $t \geq 1$, α^t is an action on $\Gamma^t = \{|\varphi\rangle^{\otimes t} : |\varphi\rangle \in \Gamma\}$

$$\alpha^t_x \ : \ |\varphi\rangle^{\otimes t} \mapsto |x \cdot \varphi\rangle^{\otimes t}$$

Example 2. Let $f \ : \ G \rightarrow S$ a hiding function,

$$|f\rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle|f(g)\rangle.$$

The y-translate of f is $y \cdot f : g \mapsto f(gy)$

$$\Gamma(f) = \{|y \cdot f\rangle : y \in G\}$$

.

The translation action on $\Gamma(f)$ is $\tau_x \ : \ |f'\rangle \mapsto |x \cdot f'\rangle$

Oracle for $f \implies$ oracle for the translation action :

$$|x\rangle|x \cdot f'\rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} |x\rangle|g\rangle|f'(gx)\rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} |x\rangle|gx^{-1}\rangle|f'(g)\rangle$$

The stabilizer of $|\varphi\rangle$ is $G_{|\varphi\rangle} = \{x \in G : |x \cdot \varphi\rangle = |\varphi\rangle\}$.

The orbit of $|\varphi\rangle$ is $G(|\varphi\rangle) = \{|x \cdot \varphi\rangle, x \in G\}$.

The orbit coset of $|\varphi_0\rangle$ and $|\varphi_1\rangle$ is $\{u \in G : |u \cdot \varphi_1\rangle = |\varphi_0\rangle\}$.

The orbit coset is empty or a left coset $uG_{|\varphi_1\rangle}$.

STABILIZER

Input: $G, \alpha, \Gamma, |\varphi\rangle$.

Output: $G_{|\varphi\rangle}$

ORBIT COSET

Input: $G, \alpha, \Gamma, |\varphi_0\rangle, |\varphi_1\rangle$.

$$\text{Output: } \begin{cases} \texttt{reject}, & \text{if } G(|\varphi_0\rangle) \cap G(|\varphi_1\rangle) = \emptyset; \\ u \in G \text{ s.t. } |u \cdot \varphi_1\rangle = |\varphi_0\rangle \text{ and generators for } G_{|\varphi_1\rangle}, & \text{ow.} \end{cases}$$

Theorem. When $t = \text{poly}(\log|G|)$ then for the translation action τ^t

- HIDDEN SUBGROUP $\leq$ STABILIZER
- HIDDEN TRANSLATION $\leq$ ORBIT COSET

Proof. The subgroup hidden by f is the stabilizer of $|f\rangle$. The translation of (f_0, f_1) is the orbit coset of $(|f_0\rangle, |f_1\rangle)$.

Theorem. Let G Abelian. When $t = \Omega(\log(|G|)$, STABILIZER for α^t is solvable in quantum time $\mathrm{poly}(\log|G|)$.

Proof. On input $|\varphi\rangle^{\otimes t}$ let $f(x) = |x \cdot \varphi\rangle$. Then f hides $G_{|\varphi\rangle}$. Run the algorithm for HIDDEN SUBGROUP, simulating the i^{th} query $|x\rangle|0\rangle_S$ using the i^{th} copy of $|\varphi\rangle$.

Theorem. Let $G = \mathbb{Z}_p^n$. When $t = \Omega(p(n+p)^{p-1})$, ORBIT COSET for α^t is solvable in quantum time $(n+p)^{O(p)}$.

Proof. One can suppose w.l.o.g. that the stabilizers of the input $|\varphi_0\rangle^{\otimes t}$, $|\varphi_1\rangle^{\otimes t}$ are trivial. Let $f_b(x) = |x \cdot \varphi_b\rangle$. Then the translation of (f_0, f_1) is the orbit coset of $(|\varphi_0\rangle, |\varphi_1\rangle)$. Run the algorithm **Translation finding**.

Idea. Let $N \lhd G$. Given PROBLEM on G, establish self-reducibility

$$\text{PROBLEM}(G) \leq \{\text{PROBLEM}(N), \text{PROBLEM}(G/N)\}$$

Definition. Orbit superposition

$$|N \cdot \varphi\rangle = \frac{1}{\sqrt{|N(|\varphi\rangle)|}} \sum_{|\varphi'\rangle \in N(|\varphi\rangle)} |\varphi'\rangle$$

Definition. Factor group action

$$\Gamma_N = \{|N \cdot \varphi\rangle : |\varphi\rangle \in \Gamma\}$$

$$\begin{array}{rcl} \alpha_N \; : \; G/N & \rightarrow & \text{Perm}(\Gamma_N) \\ xN & \mapsto & \alpha_{N,x} \end{array}$$

$$\alpha_{N,x}(|N \cdot \varphi\rangle) = |x \cdot (N \cdot \varphi)\rangle$$

How to create the orbit superposition $|N \cdot \varphi\rangle$?

Theorem. G solvable. Given $|\varphi\rangle^{\otimes(s+\lfloor\log|G|\rfloor+1)}$, realizing $|\varphi\rangle|G\cdot\varphi\rangle^{\otimes s}$ is reducible to ORBIT COSET in subgroups of G for α.

Proof. Let $G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_m = \{1_G\}$ where G_n/G_{n+1} is cyclic of prime order r_n and is generated by $z_n G_{n+1}$.

For $n = m$ downto 0, produce the state $|\varphi\rangle|G_n \cdot \varphi\rangle^{\otimes(s+n)}$.

Induction step. Set $k = s+n+1$. Given $|\varphi\rangle|G_{n+1}\cdot\varphi\rangle^{\otimes k}$

- Compute k copies of $\frac{1}{\sqrt{r_n}}\sum_{i=0}^{r_n-1}|i\rangle|z_n^i\cdot(G_{n+1}\cdot\varphi)\rangle$
- Disentangle the first registers by the method of Watrous

QFT: $(\frac{1}{\sqrt{r_n}}\sum_{j=0}^{r_n-1}|j\rangle|\psi_j\rangle)^{\otimes k}$ where $|\psi_j\rangle = \frac{1}{\sqrt{r_n}}\sum_{i=0}^{r_n-1}\omega_{r_n}^{ij}|z_n^i\cdot(G_{n+1}\cdot\varphi)\rangle$.

Suppose $j_0 \neq 0$. Then $|(z_n^i g)^{jj_0^{-1}}\cdot\psi_{j_0}\rangle = \omega_{r_n}^{-ij}|\psi_{j_0}\rangle$ for $g \in G_{n+1}$.

ORBIT COSET on $|\varphi\rangle$ and $|z_n^i g\cdot\varphi\rangle$ gives $z_n^i g$.

Theorem. Let $N \lhd G$, N solvable. When $t = \Omega(s + \log|G|)$

- OC(G, α^t) $\leq$ $\{$ OC(Subgroups of N, α), OC$(G/N, (\alpha_N)^s)$ $\}$
- STAB(G, α^t) $\leq$ $\{$ OC(Subgroups of N, α), STAB$(G/N, (\alpha_N)^s)$ $\}$

Proof for STABILIZER.

Compute $N_{|\varphi\rangle} = G_{|\varphi\rangle} \cap N$ by STAB(N, α).

Construct $H \leq G$ such that

$$N_{|\varphi\rangle} \leq H \leq G_{|\varphi\rangle} \text{ and } HN/N = G_{|\varphi\rangle}N/N.$$

Then $H = G_{|\varphi\rangle}$ since $H \cap N = G_{|\varphi\rangle} \cap N$ and $HN/N = G_{|\varphi\rangle}N/N$.

Add to $N_{|\varphi\rangle}$ generators of $G_{|\varphi\rangle}N/N$ which are in $G_{|\varphi\rangle}$

Fact. $G_{|\varphi\rangle}N/N$ is the stabilizer of $|N \cdot \varphi\rangle$ in G/N.

- Compute V such that $\langle V \rangle = G_{|\varphi\rangle}N/N$ by STAB$(G/N, (\alpha_N)^s)$
- Create input $|N \cdot \varphi\rangle^{\otimes s}$ by OC(N, α)
- Let $z \in V$. Then $z = gn^{-1}$ for $g \in G_{|\varphi\rangle}$ and $n \in N$. In N the orbit coset of $|z^{-1}\varphi\rangle$ and $|\varphi\rangle$ is $nN_{|\varphi\rangle}$. Find n by OC(N, α).

Definition. A solvable group is smoothly solvable if it is of bounded exponent and its derived series is of bounded length.

Fact. A smoothly solvable group G has a smooth series

$$G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_m = \{1_G\}$$

where m bounded and G_i/G_{i+1} is elementary Abelian of bounded exponent.

Theorem. Let G smoothly solvable. When $t = \log^{\Omega(1)}|G|$ then ORBIT COSET can be solved for α^t in quantum time $\mathrm{poly}(\log|G|)$.

Theorem. Let G solvable such that G' is smoothly solvable. When $t = \log^{\Omega(1)}|G|$ then STABILIZER can be solved for α^t in quantum time $\mathrm{poly}(\log|G|)$.

Corollary There is a quantum polynomial time algorithm for

- HIDDEN TRANSLATION in smoothly solvable groups
- HIDDEN SUBGROUP in solvable groups having a smoothly solvable commutator subgroup