Entanglement in quantum critical phenomena

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Entanglement in quantum critical phenomena

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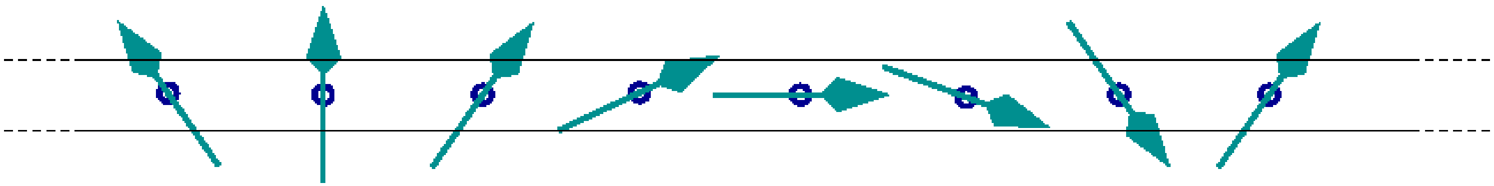
Overview

Vidal, Latorre, Rico, Kitaev, quant-ph/0211074

1. 1D Spin models and quantum phase transitions
2. Entanglement in spin chains
   - *Definitions*
   - *Computation*
3. Critical and non-critical entanglement
   - Breakdown of DMRG techniques
4. Connection with conformal field theory
   - *Entanglement in 2D and 3D spin models*
   - Monotonicity of entanglement along RG flow

1D spin models



- ***XY model*** with magnetic field

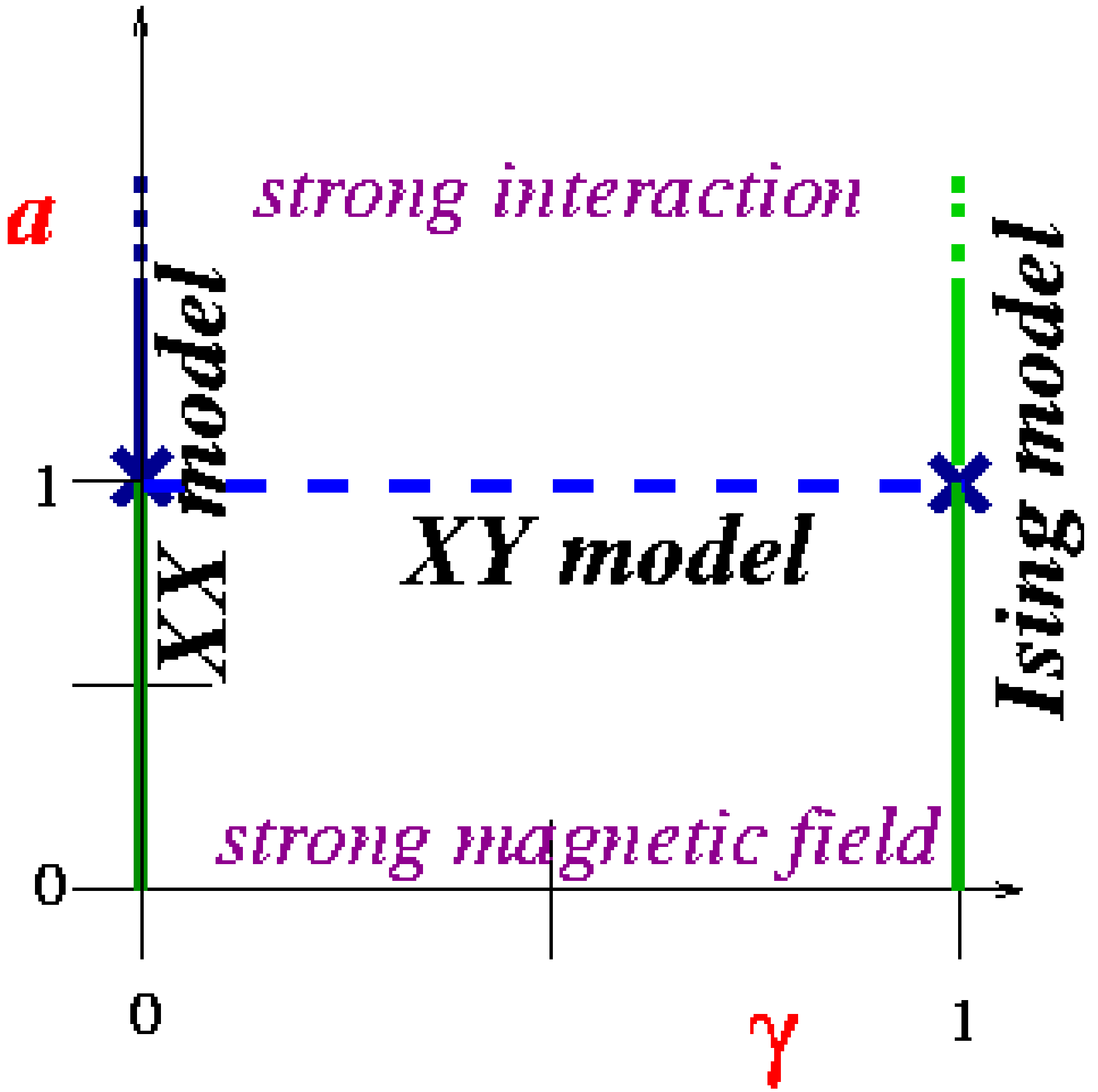
  [including ***XX model*** and ***Ising model***]

$$H_{XY}=-\sum_{l=0}^{N-1}\left(\frac{a}{2}\left[(1+\gamma)\sigma_l^x\sigma_{l+1}^x+(1-\gamma)\sigma_l^y\sigma_{l+1}^y\right]+\sigma_l^z\right)$$

- ***XXZ model*** with magnetic field

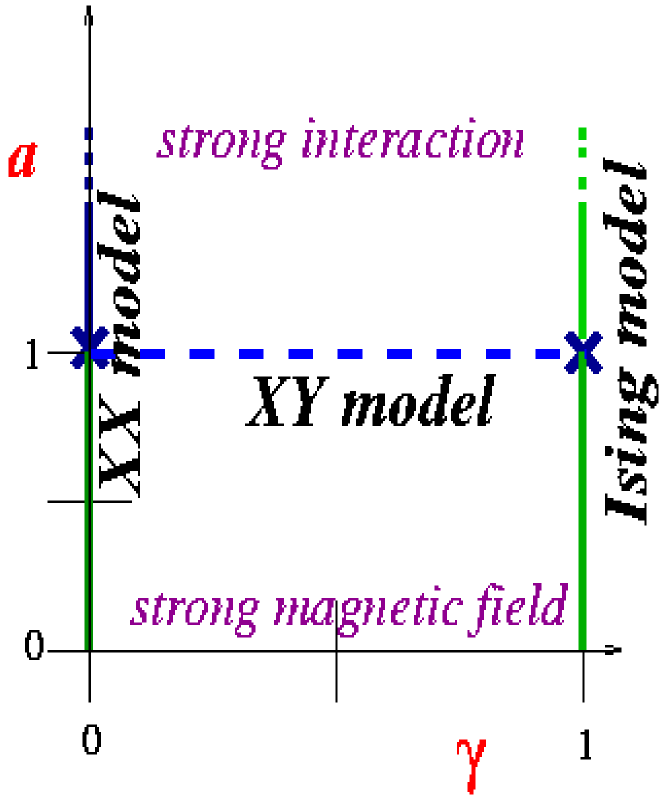
$$H_{XY}=-\sum_{l=0}^{N-1}\left(\sigma_l^x\sigma_{l+1}^x+\sigma_l^y\sigma_{l+1}^y+\Delta\sigma_l^z\sigma_{l+1}^z+\lambda\sigma_l^z\right)$$

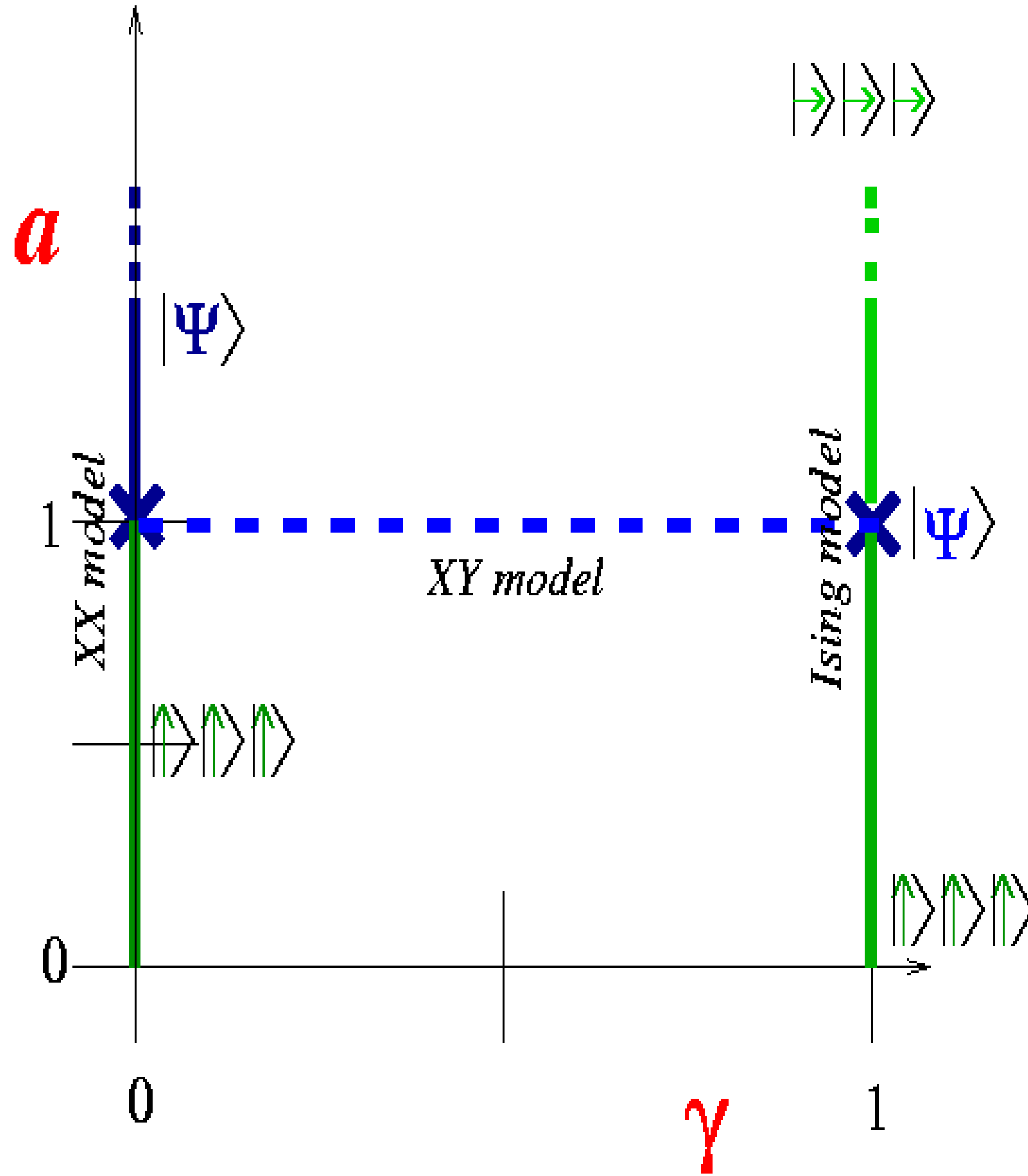
XY model



a
strong interaction
XX model
Ising model
1
XY model
strong magnetic field
0
0
γ
1

Entanglement and phases





Quantum phase transition

- T=0
- $H = H_0 + g H_1$

non-analiticity of ground-state energy

  - Level crossing, $[H_0 , H_1]=0$ (finite chain)
  - Thermodynamic limit (infinite chain)

- Long-range correlations

$$\langle \sigma_l \sigma_{l+d} \rangle \propto \frac{1}{d^p}$$

Previous work

- Dorit Aharonov, Phys. Rev. A 62 (2000);

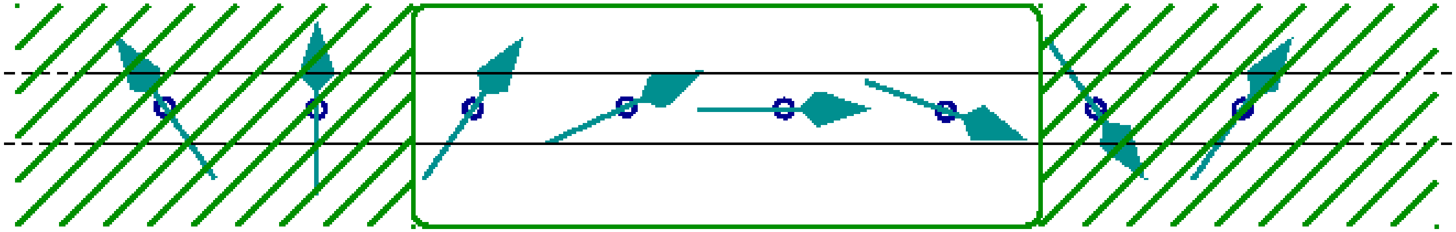
Transition from quantum to classical in noisy quantum computers.

- Osborne and Nielsen [quant-ph/0109024; quant-ph/0202162]
- Osterloh, Amico, Falci and Falzio, Nature 416 (2002)

Single spin and two-spin entanglement measures have a peak at or close to a phase transition

Entanglement in a spin chain

- We measure the entanglement between a block ***B*** of spins and the rest of the chain



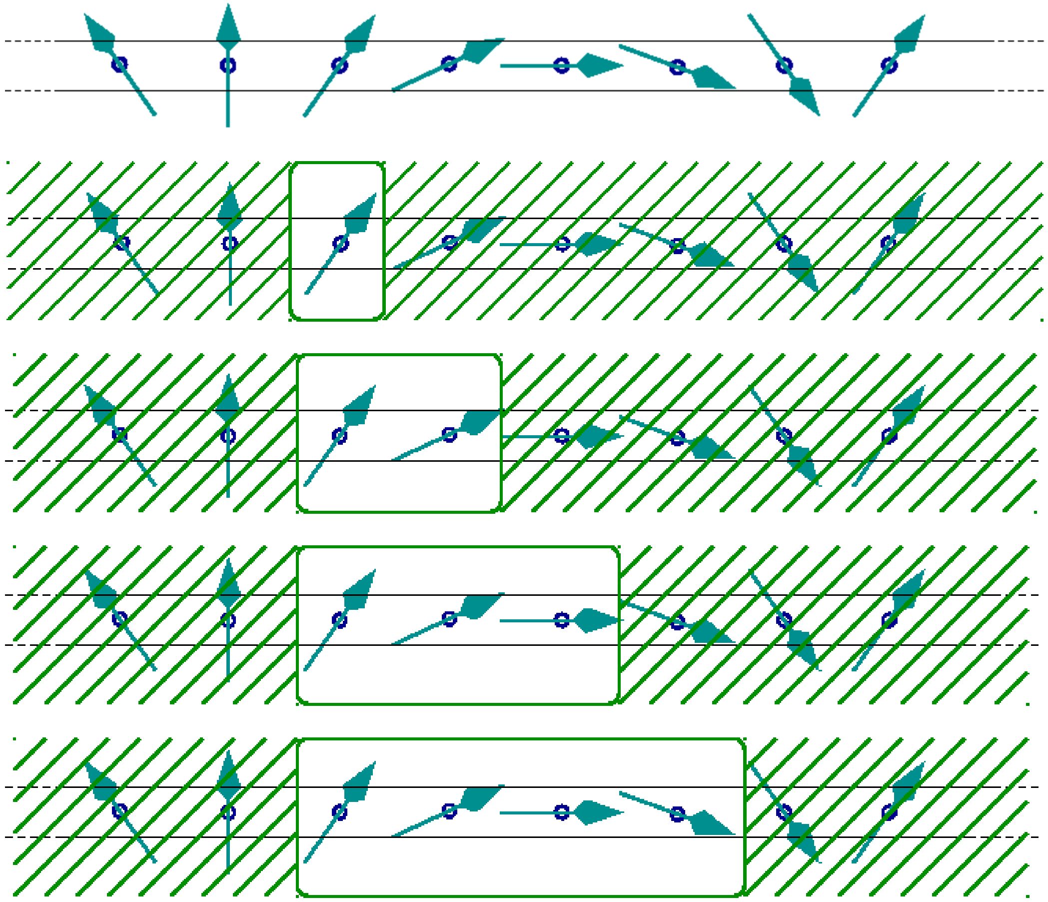
$$\rho_B = tr_{chain-B} \left| \Psi_g \right\rangle \left\langle \Psi_g \right|$$

Entropy of entanglement

$$S_B = -tr\left(\rho_B \log \rho_B\right)$$

Bennett, Bernstein, Popescu and Schumacher, PRA 53 (1996)

Entropy of entanglement



Osborne and Nielsen, quant-ph/0202162

$$\rho_1 \to S_1 = S(\rho_1)$$

$$\rho_2 \to S_2 = S(\rho_2)$$

$$\rho_3 \to S_3 = S(\rho_3)$$

$$\rho_4 \to S_4 = S(\rho_4)$$

……

Computation of entanglement: infinite XY spin chain

- Change of variables

$$c_{2l} = \left(\prod_{m=0}^{l-1} \sigma_m^z \right) \sigma_l^x \qquad c_{2l+1} = \left(\prod_{m=0}^{l-1} \sigma_m^z \right) \sigma_l^y$$

Majorana operators

$$c_m^+ = c_m$$

$$\{c_m, c_n\} = 2\delta_{mn}$$

$$H_{XY}(\{\sigma_l^\alpha\}) \to H_{XY}(\{c_l\})$$

- The ground state $\left|\Psi_g\right\rangle$ is ***gaussian***
- Compute *correlation matrix*

$$\left\langle c_m c_n \right\rangle = \delta_{mn} + iB_{mn} \qquad m,n = 0,1,\ldots,2N-1$$

$$B = \begin{bmatrix} \Pi_0 & \Pi_1 & \cdots & \Pi_{N-1} \\ \Pi_{-1} & \Pi_0 & & \vdots \\ \vdots & & \ddots & \vdots \\ \Pi_{1-N} & \cdots & \cdots & \Pi_0 \end{bmatrix} \qquad \Pi_0 = \begin{bmatrix} 0 & g_l \\ -g_{-l} & 0 \end{bmatrix}$$

$$g_l = \tfrac{1}{2\pi}\int_0^{2\pi} d\phi \exp(-il\phi) \frac{a\cos\phi - 1 - ia\gamma\sin\phi}{|a\cos\phi - 1 - ia\gamma\sin\phi|}$$

- The state ρ_L is also ***gaussian***
- Compute *correlation matrix*

$$\langle c_m c_n \rangle = \delta_{mn} + i(B_L)_{mn} \quad m,n = 0,1,\ldots,2L-1$$

$$B = \begin{bmatrix} \Pi_0 & \Pi_1 & \cdots & \cdots & \Pi_{N-1} \\ \Pi_{-1} & \Pi_0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ \Pi_{1-N} & \cdots & \cdots & \cdots & \Pi_0 \end{bmatrix} \rightarrow B_L = \begin{bmatrix} \Pi_0 & \Pi_1 & \cdots & \Pi_{L-1} \\ \Pi_{-1} & \Pi_0 & & \vdots \\ \vdots & & \ddots & \vdots \\ \Pi_{1-L} & \cdots & \cdots & \Pi_0 \end{bmatrix}$$

- Diagonalize *correlation matrix*

$$B_L' = VB_LV^+ = \begin{bmatrix} \nu_1\Pi & & & \\ & \nu_2\Pi & & \\ & & \ddots & \\ & & & \nu_L\Pi \end{bmatrix} \qquad \Pi = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

New majorana operators

$$\{c_m\} \xrightarrow{V} \{d_m\}$$

$$d_m^+ = d_m$$

$$\{d_m, d_n\} = 2\delta_{mn}$$

$$\langle d_m d_n \rangle = \delta_{mn} + i(B_L')_{mn}$$

- Introduce *fermionic operators*

$$a_l = \frac{1}{2}\left(d_{2l} + i d_{2l+1}\right)$$

$$\{a_m, a_n\} = \{a_m^+, a_n^+\} = 0$$
$$\{a_m^+, a_n\} = \delta_{mn}$$

Their *correlation matrix* is

$$\langle b_m b_n \rangle = 0$$
$$\langle b_m^+ b_n \rangle = \delta_{mn} \frac{1+\upsilon_m}{2}$$

Therefore the L fermionic modes are *uncorrelated*

$$\rho_L = \sigma_1 \otimes \sigma_2 \otimes \ldots \otimes \sigma_L$$

- Therefore the entropy of mode *m* is

$$S(\sigma_m) = H_2\left(\frac{1+\nu_m}{2}\right)$$

- And the entropy of the L spins reads

$$S_L = \sum_{m=1}^{L} H_2\left(\frac{1+\nu_m}{2}\right)$$

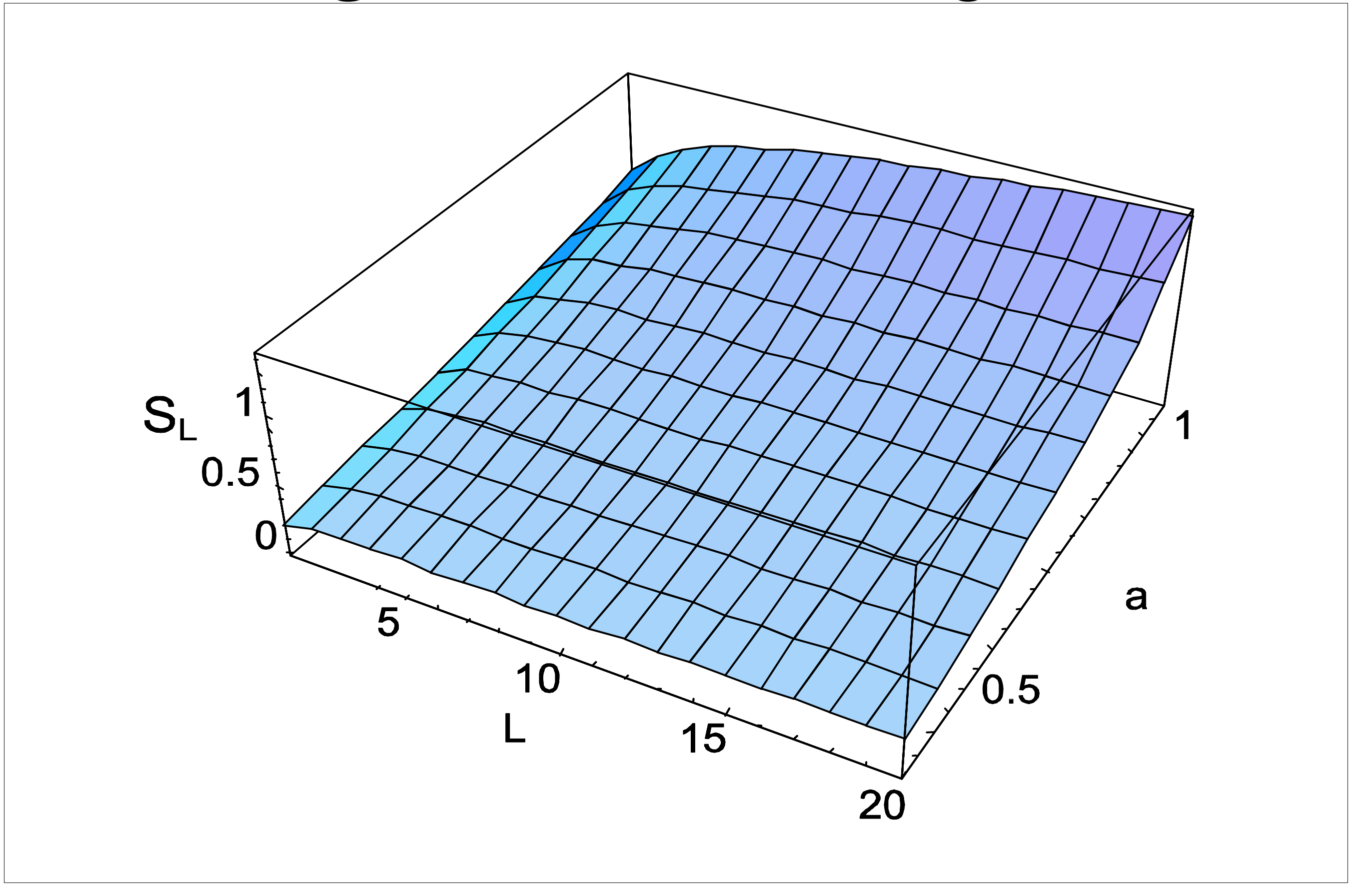
Computation of entanglement : finite XXZ spin chain

- Bethe ansatz for N=20 spins

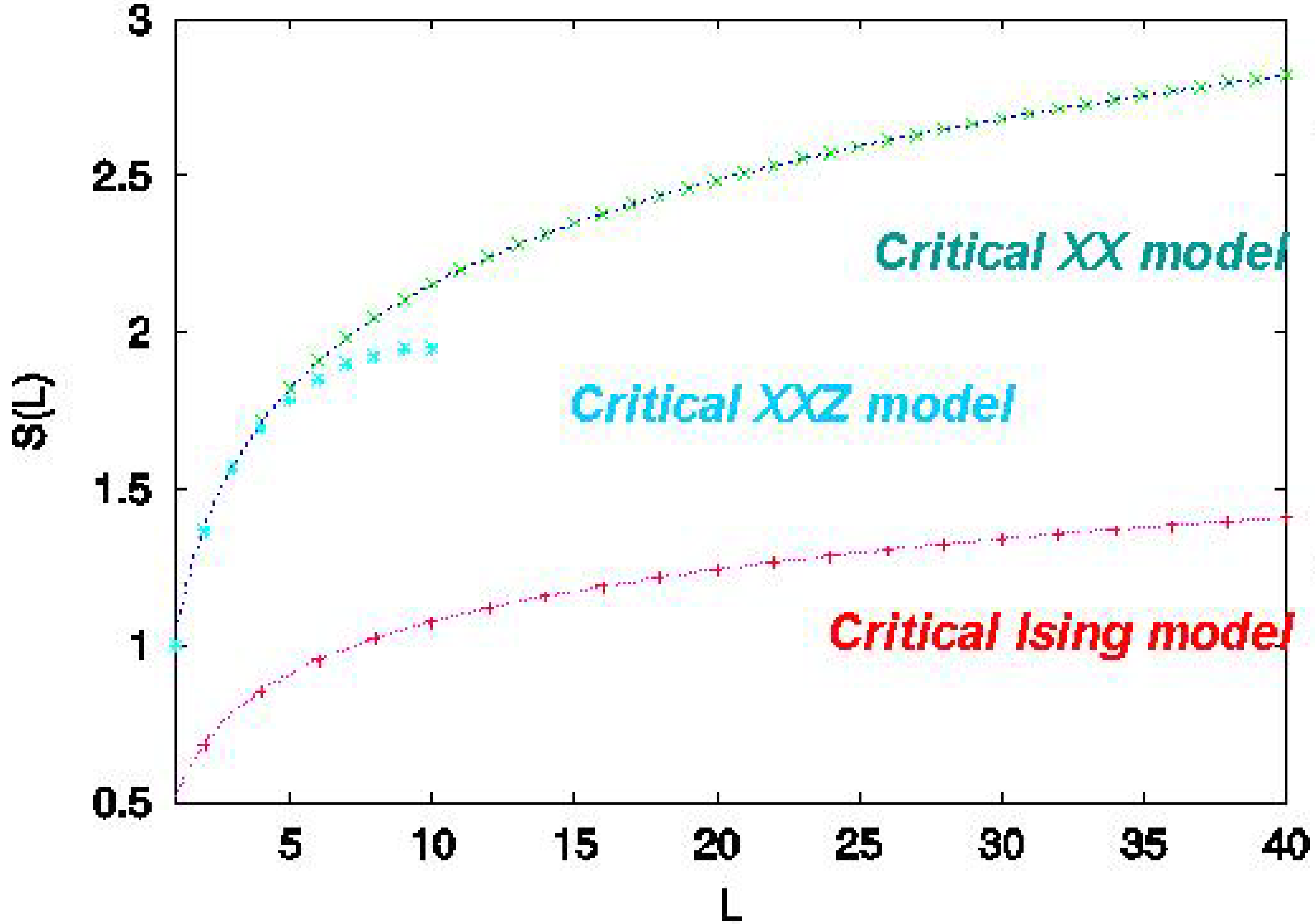
$$\left|\Psi_g\right\rangle \rightarrow \rho_L \rightarrow S_L$$

Non-critical entanglement

Ising chain with magnetic field



Critical entanglement



$$S_L \approx \frac{1}{3} \log L$$

$$S_L \approx \frac{1}{6} \log L$$

Non-critical versus critical ground-state entanglement

- *Non-critical* entanglement has a *saturation value*

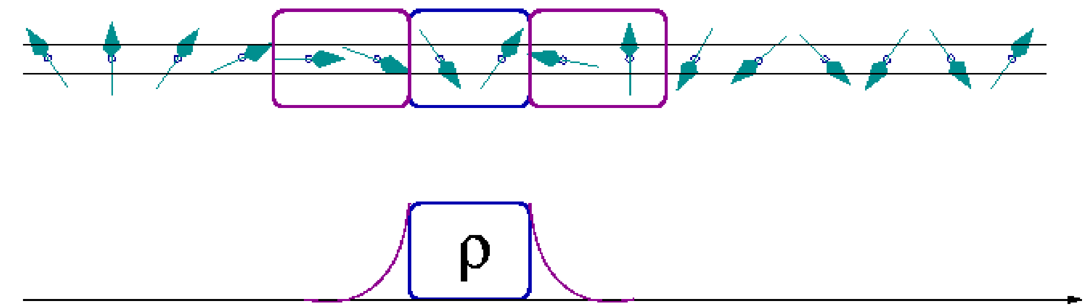
$$S_L \leq S_{\max}$$

- *Critical* entanglement diverges logarithmically with the number L of spins

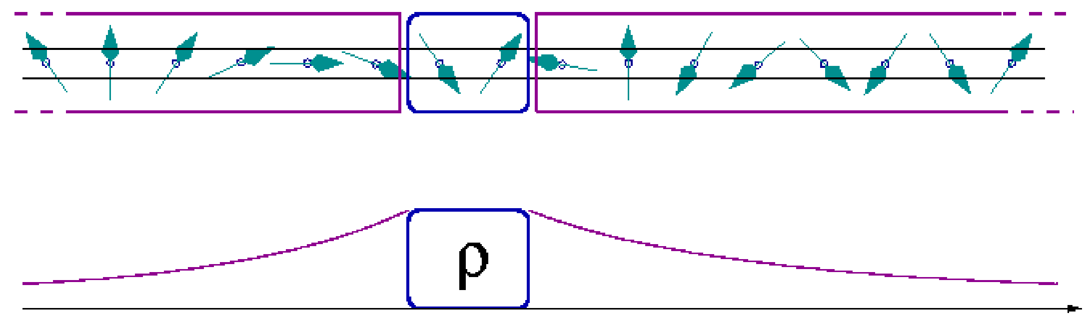
$$S_L \approx k \log L$$

Density Matrix Renormalization Group Techniques

Non-critical entanglement is ***semi-local***: The state of L qubits can be reconstructed by considering a few neighbors → DMRG



Critical entanglement embraces the system at ***all length scales***. It is not possible to construct ρ_L locally (its rank diverges) by DMRG



Connection to conformal field theory

- Geometric or fine-grained entropy

$$S_L \approx \frac{c+\bar{c}}{6} \log L$$

Holzhey, Larsen, Wilczek, Nucl. Phys. B 424 (1994)

Srednicki, PRL 71 (1993)

Fiola, Preskill, Strominger, Trivedi, PRD 50 (1994)

c is the ***central charge*** of the theory (holomorphic and antiholomorphic central charges)

$c_b = \bar{c}_b = 1$ for a *free boson* (XX model)

$c_f = \bar{c}_f = ½$ for a *free fermion* (Ising model)

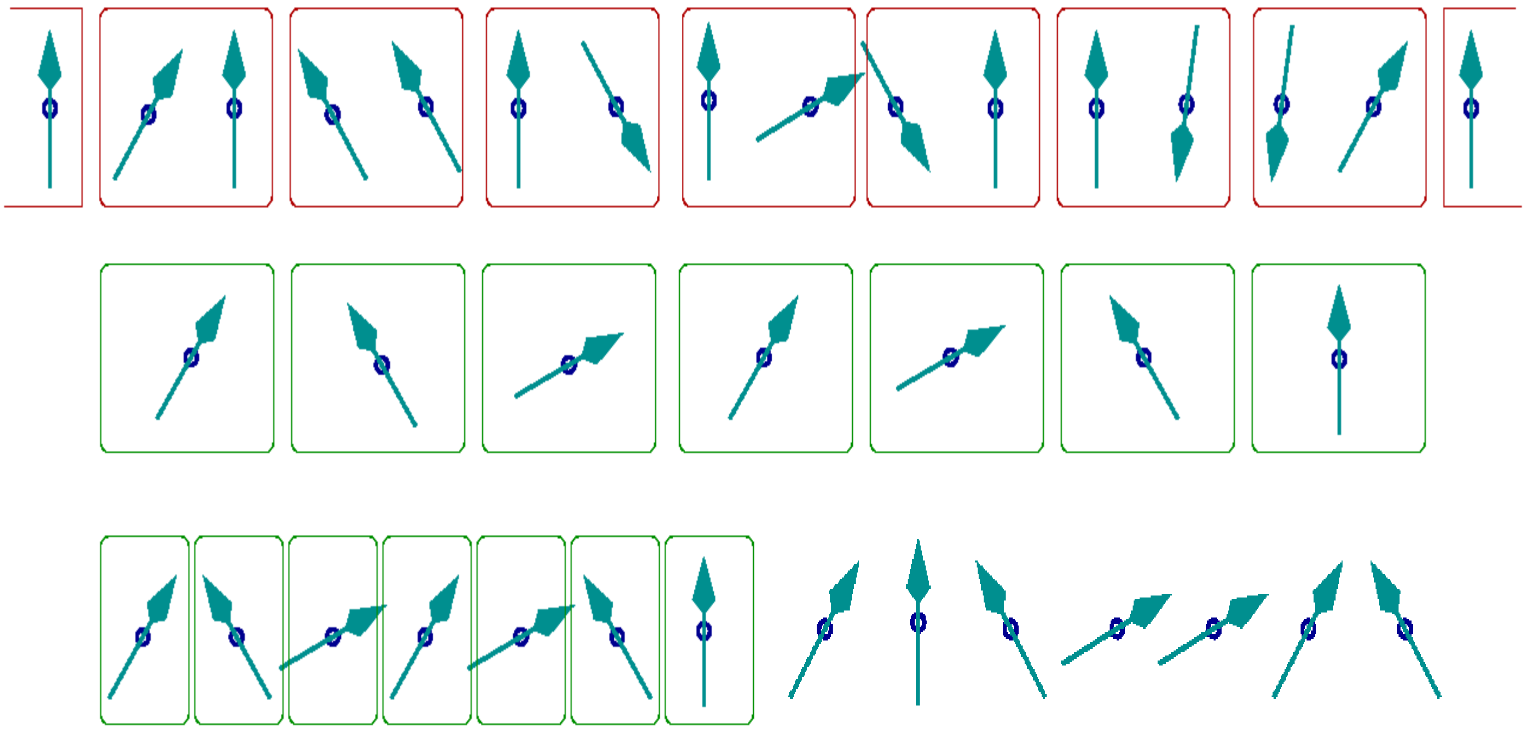
Entanglement in 2D and 3D

- We can export a result by **Srednicki, PRL 71 (1993)**

$$S_R \approx \kappa \Sigma(R)$$

The entanglement of a region *R* grows proportional to the size $\Sigma(R)$ of the boundary of *R*.

Renormalization Group flow

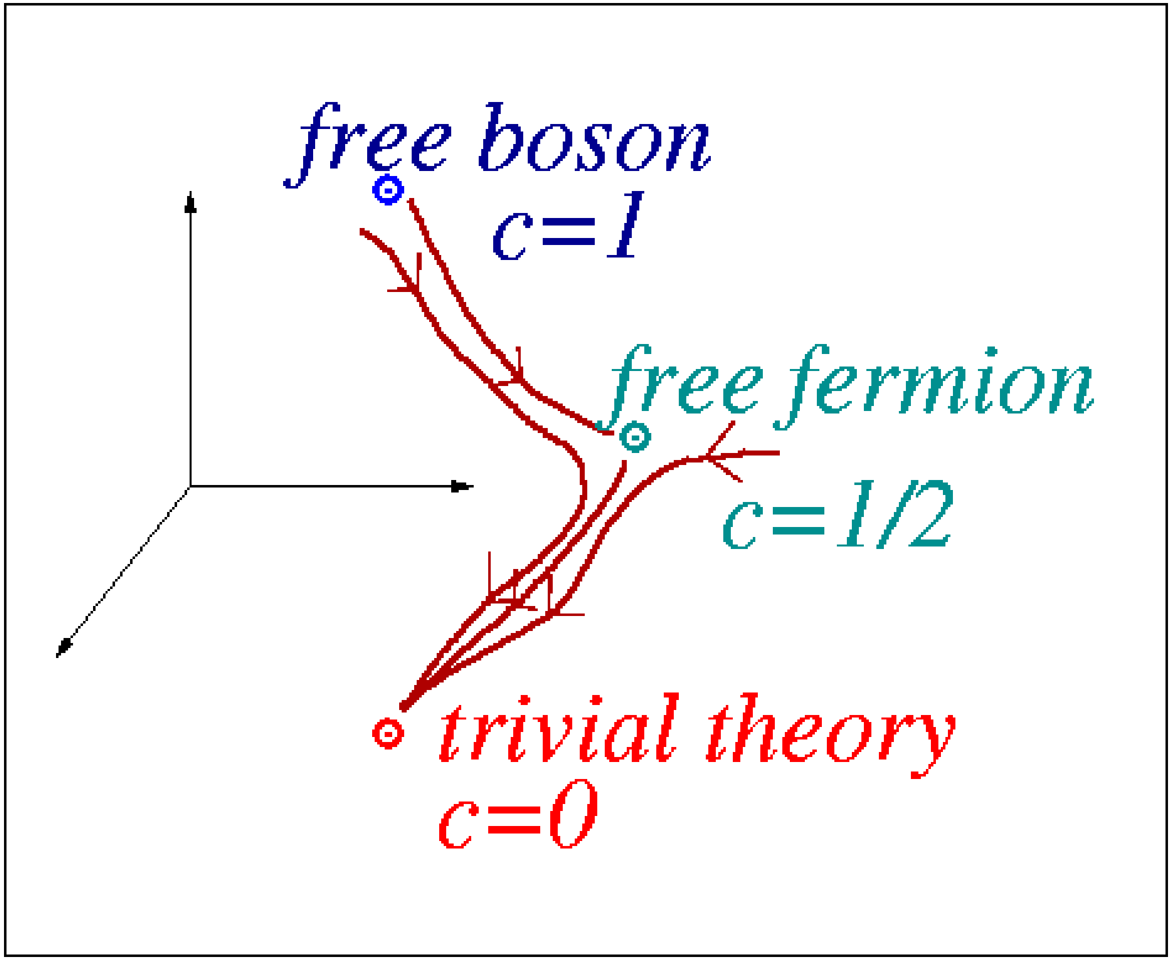


c-theorem

- The c-theorem establishes that the *central charge* can only *decrease* along the renormalization group flow

Zamolodchikov, JETP Lett 43 (1986)
Capelli, Friedan, Latorre, Nucl. Phys. B 352 (1991)
Forte, Latorre, Nucl. Phys. B 535 (1998)

Renormalization Group flow and entanglement



Monotonicity of entanglement

- Under LOCC (local manipulation of a composite system)
- Along RG flow (change of scale)

Conclusions

Computation of entanglement in several 1D ***critical*** and ***non-critical*** spin models

-*Two distinctive forms of entanglement*

→Breakdown of DMRG techniques

-*Connection to conformal field theory*

→2D and 3D entanglement

→Monotonicity under RG transformations