New developments of rigorous Feynman path integrals and application to a stochastic Schröedinger equation

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(joint work with S. Albeverio and G. Guatteri)

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi + V\psi \\ \psi(0,x) = \phi(x) \end{cases}$$

$$\psi(t,x) = \text{“} \quad const \int_{\{\gamma|\gamma(t)=x\}} e^{\frac{i}{\hbar}S_t(\gamma)}\phi(\gamma(0))D\gamma \quad \text{''}$$

Feynman heuristic measure $e^{\frac{i}{\hbar}S_t(\gamma)}D\gamma$ cannot be $\sigma-$ additive and of bounded variation

$\Downarrow$

It can be realized as a linear continuous functional on a Banach algebra of functions

Infinite dimensional oscillatory integrals [Ito, Albeverio and Høegh-Krohn, Elworthy and Truman, Albeverio and Brzeźniak]

- classical results: definition and applications of "∞-dimensional Fresnel integrals"
- New developments:
  - Phase space Feynman path integrals $\to$ Schrödinger equation with momentum-dependent potential
  - Feynman path integrals with complex phase $\to$ stochastic Schrödinger equation.

Finite and infinite dimensional Fresnel integrals on Hilbert spaces

$$x \in \mathcal{H}, \quad A : \mathcal{H} \to \mathcal{H}, \quad f : \mathcal{H} \to \mathbb{C}$$

$$\int_{\mathcal{H}} e^{\frac{i}{2\hbar}\langle x, Ax\rangle} f(x) dx$$

$\mathcal{H}$ $n-$ dimensional: $\phi \in S(\mathbb{R}^n)$, $\phi(0) = 1$

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle x, Ax\rangle} \phi(\epsilon x) f(x) dx$$

$$\equiv \widetilde{\int_{\mathbb{R}^n}} e^{\frac{i}{2\hbar}\langle x, Ax\rangle} f(x) dx$$

$\mathcal{H}$ $\infty-$ dimensional: $P_n \uparrow I_{\mathcal{H}}$

$$\lim_{n \to \infty} (2\pi\hbar)^{n/2} \int_{P_n\mathcal{H}} e^{\frac{i}{2\hbar}\langle P_n x, AP_n x\rangle} f(P_n x) dP_n x$$

$$\equiv \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x, Ax\rangle} f(x) dx$$

Cameron-Martin type formula:

$f = \int_{\mathcal{H}} e^{i\langle k, x\rangle} \mu_f(dk)$, A invertible, $(A - I)$ trace class

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x, Ax\rangle} f(x) dx = (\det A)^{-1/2} \int_{\mathcal{H}} e^{-\frac{i\hbar}{2}\langle x, A^{-1}x\rangle} \mu_f(dk)$$

Representation of the solution of the Schrödinger equation

$$H_t = \{\gamma : [0,t] \to \mathbb{R}^d, \gamma(t) = 0, \int_0^t \dot{\gamma}(s)^2 ds < \infty\}$$

$$\langle \gamma_1, \gamma_2 \rangle = \int_0^t \dot{\gamma}_1(s)\dot{\gamma}_2(s) ds$$

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi + (\frac{\omega^2}{2}x^2 + V(x))\psi \\ \psi(0,x) = \phi(x) \end{cases} \qquad (1)$$

Theorem 1. *[Albeverio and Brzeźniak, Elworthy and Truman]*
If ϕ and V are Fourier transform of complex bounded variation measures on $\mathbb{R}^d$, then the solution of the Schrödinger equation (1) is represented by the infinite dimensional Fresnel integral:

$$\psi(t,0) = \widetilde{\int_{H_t}} e^{\frac{i}{2\hbar}\langle \gamma, (I-L)\gamma \rangle} e^{-\frac{i}{\hbar}\int_0^t V(\gamma(s))ds} \phi(\gamma(0)) d\gamma$$

where $\langle \gamma, L\gamma \rangle = \omega^2 \int_0^t \gamma^2(s) ds$

Phase space Feynman path integrals

$$\psi(t,x) = \text{“} \int_{\{(q,p)|q(t)=x\}} e^{\frac{i}{\hbar}S_t(q,p)}\phi(q(0))DqDp \text{”}$$

$$S(q,p) = \int_0^t [p(s)\dot{q}(s) - H(q(s),p(s))]ds$$

Realization as ∞-dim Fresnel integral:

$\mathcal{H} = H_t \times L_t$, H_t Cameron-Martin space, $L_t = L^2([0,t])$
$\langle (q_1,p_1),(q_2,p_2)\rangle = \int_0^t \dot{q}_1(s)\dot{q}_2(s)ds + \int_0^t p_1(s)p_2(s)ds$

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi = H\psi & H(q,p) = \frac{p^2}{2m} + V_1(q) + V_2(p) \\ \psi(0,x) = \phi(x) \end{cases} \tag{2}$$

Theorem 2. *[Albeverio, Guatteri, Mazzucchi]*
Let $V_1, \phi \in \mathcal{F}(\mathbb{R}^d)$, and $\int_0^t V_2(p(s))ds \in \mathcal{F}(L_t)$. Then the solution of the Schrödinger equation (2) can be represented by the following "phase space Feynman path integral":

$$\psi(t,0) = \widetilde{\int_{H_t\times L_t}} e^{\frac{i}{2\hbar}\langle (q,p),A(q,p)\rangle} e^{-\frac{i}{\hbar}\int_0^t V_1(q(s))ds} e^{-\frac{i}{\hbar}\int_0^t V_2(p(s))ds}\phi(q(0))dqdp$$

with $\langle (q,p),A(q,p)\rangle = 2\int_0^t [p(s)\dot{q}(s) - \frac{p(s)^2}{2}]ds$

Oscillatory integrals with complex phase

Theorem 3. *[Albeverio, Guatteri, Mazzucchi]*
Let L_1 and L_2 be commuting simmetric trace class operators in a real Hilbert space $\mathcal{H}$, such that L_2 is nonnegative and $I + L_1$ is invertible. Let $l \in \mathcal{H}$ and $f : \mathcal{H} \to \mathbb{C}$ be the Fourier transform of a finite complex Borel measure on $\mathcal{H}$:

$$f(x) = \hat{\mu}(\gamma), \qquad f(x) = \int_{\mathcal{H}} e^{i\langle x,y\rangle} d\mu_f(y).$$

Then infinite-dimensional oscillatory integral

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I+L_1)x\rangle} e^{-\frac{1}{2\hbar}\langle x,L_2 x\rangle+\langle l,x\rangle} f(x)dx$$

is well defined and it is given by:

$$\det(I+L)^{-1/2} \int_{\mathcal{H}} e^{\frac{-i\hbar}{2}\langle x-il,(I+L)^{-1}x-il\rangle} \mu_f(dx)$$

where $L = L_1 + iL_2$

Belavkin equation: a stochastic Schrödinger equation

V.P. Belavkin. *A new wave equation for a continuous nondemolition measurement.* *Phys. Lett.*, vol. A140, no.7/8: 355-358, 1989.

$$d\psi = -\frac{i}{\hbar}H\psi dt - \frac{\lambda x^2}{2}\psi dt + \sqrt{\lambda}x\psi dW(t)$$

Physical meaning of the solution:

$(C_t, |\cdot|, P), \qquad I \in \mathcal{B}$

$$\mathbb{P}(X(s) = \omega(s)_{s\in[0,t]} \in I) = \int_I \|\psi(t,\omega)\|^2 P(d\omega),$$

$$\mathbb{E}(Z(t)|X(s) = \omega(s)_{s\in[0,t]} \in I) = \int_I \frac{\langle\psi(t,\omega), Z\psi(t,\omega)\rangle}{\|\psi(t,\omega)\|^2} P(d\omega).$$

Continuous measurement of an observable A:

$$d\psi = -\frac{i}{\hbar}H\psi dt - \frac{\lambda A^2}{2}\psi dt + \sqrt{\lambda}A\psi dW(t)$$

Continuous measurement of momentum p:

$$d\psi = -\frac{i}{\hbar}H\psi dt - \frac{\lambda p^2}{2}\psi dt + \sqrt{\lambda}p\psi dW(t)$$

Solution via Feynman path integrals with complex pha:

M.B. Mensky: "restricted path integrals"

$$\psi(t,x,[a]) = \text{“} \int_{\{\gamma(t)=x\}} e^{\frac{i}{\hbar}S_t(\gamma)} e^{-k\int_0^t(\gamma(s)-a(s))^2ds}\phi(\gamma(0))D\gamma \text{”}$$

Theorem 4. *[Albeverio, Guatteri, Mazzucchi] Let V and ϕ be Fourier transform of finite complex measures on $\mathbb{R}^d$. Then there exist a solution of the stochastic Schrödinger equation*

$$\begin{cases} d\psi = -\frac{i}{\hbar}H\psi dt - \frac{\lambda x^2}{2}\psi dt + \sqrt{\lambda}x\psi dW(t) \\ \psi(0,x) = \phi(x) \qquad t \geq 0,\ x \in \mathbb{R} \end{cases}$$

and it can be represented by the following infinite dimensional oscillatory integral with complex phase on the Cameron Martin space H_t:

$$\psi(t,x) = \widetilde{\int} e^{\frac{i}{\hbar}S_t(\gamma)-\lambda\int_0^t(\gamma(s)+x)^2ds}$$
$$e^{\int_0^t\sqrt{\lambda}(\gamma(s)+x)dW(s)ds}\phi(\gamma(0)+x)d\gamma$$

$$= e^{-\lambda|x|^2t+\sqrt{\lambda}x\cdot\omega(t)}\widetilde{\int_H} e^{\frac{i}{2\hbar}\langle\gamma,(I+L)\gamma\rangle}e^{\langle l,\gamma\rangle}e^{-2\lambda\hbar\int_0^t x\cdot\gamma(s)ds}$$
$$\cdot e^{-i\int_0^t V(x+\gamma(s))ds}\phi(\gamma(0)+x)d\gamma$$

where $l \in H_t$, $l(s) = \sqrt{\lambda}\int_s^t\omega(\tau)d\tau$ and $L : H_t^{\mathbb{C}} \to H_t^{\mathbb{C}}$, $\langle\gamma_1, L\gamma_2\rangle = -2i\lambda\hbar\int_0^t\gamma_1(s)\gamma_2(s)ds$

Momentum measurement

$$\begin{cases} d\psi = -\frac{i}{\hbar}H\psi dt - \frac{\lambda p^2}{2}\psi dt + \sqrt{\lambda}p\psi dW(t) \\ \psi(0,x) = \phi(x) \qquad t \geq 0,\ x \in \mathbb{R} \end{cases} \tag{2}$$

$$\psi(t,x) = \widetilde{\int} \exp\left(\frac{i}{\hbar}S(q+x,p) - \lambda \int_0^t p^2(s)ds\right) \cdot$$
$$\exp\left(\sqrt{\lambda}\int_0^t p(s)dW(s)\right)\phi(q(0)+x)dqdp$$

Theorem 5. *Let V and ϕ be Fourier transform of finite complex measures on $\mathbb{R}^d$. Then there exist a solution of the Cauchy problem (2), which can be represented by the following "phase space Feynman path integral":*

$$\psi(t,0,\omega) = \widetilde{\int_{H_t \times L_t}} e^{\frac{i}{\hbar}\int_0^t (p(s)\dot{q}(s) - \frac{p^2(s)}{2})ds} e^{-\lambda \int_0^t p(s)^2 ds}$$
$$e^{-\frac{i}{\hbar}\int_0^t V(q(s))ds} e^{\sqrt{\lambda}\int_0^t p(s)dW(s)} \phi(q(0))dqdp \tag{3}$$

$$= \widetilde{\int_{H_t \times L_t}} e^{\frac{i}{\hbar}\langle (q,p), A(q,p)\rangle} e^{\langle (q,p), l\rangle}$$
$$e^{-\frac{i}{\hbar}\int_0^t V(q(s))ds} \phi(q(0))dqdp \tag{4}$$

where $\langle (q,p), A(q,p)\rangle = 2\int_0^t (p(s)\dot{q}(s)ds - (1-2i\lambda\hbar)\int_0^t p^2(s))ds$ *and* $l \in H_t \times L_t$.

Further developments

- Representation of the solution of a large class of Belavkin equation, describing continuous measurement of different observables

- Semiclassical approximation of the solution: limit $\hbar \to 0$

- Rigorous mathematical definition of Feynman path integral when the potential V has polynomial growth.