Time Dependent Resonances

Resonances are poles of the S-matrix, or Green's function, Gamow states, other outgoing solutions, quasimodes (semiclassical or other...)...

In a time dep. picture, we observe exponential decay: $e^{-\Gamma t}$

The Goal: Time Dep. Theory

We start with the math problem of perturbations of embedded e.values:

Given a self adjoint operator H_0, with cont. spec. and e.value λ_0 in the cont. spec., with normalized e.vector ψ_0,

Find $\quad e^{-iHt}\psi_0 = e^{-\Gamma t}(\psi_0 + o(1)) + O(t^{-m})$

where $\quad H = H_0 + \varepsilon W_\epsilon$

The Generic Example:

$$H_0 = \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta + x^2 \end{pmatrix} \qquad W = \begin{pmatrix} 0 & w(x) \\ w(x) & 0 \end{pmatrix}$$

More generally: $H_0 = \begin{pmatrix} K_0 & 0 \\ 0 & K_p \end{pmatrix}$

where H_0 – has cont. spec. (e.g. photon, phonon field, your favorite bath)

K_p - has pure point spec.

Problem

Given initial data around e.state of H_0, find e^{-iHt}.

To this end, we need to localize the spectral support of initial data in H:

$$e^{-iHt} g_\Delta(H)\psi = ?$$

Δ interval in cont. spec. of H_0.

Theorem 1

Let H_0 satisfy conditions (H) and $\varepsilon W_\varepsilon$ satisfy conditions (W). Then

a) $H = H_0 + \varepsilon W_\varepsilon$ has no emb. e.values in Δ.

b) spec(H) is abs. cont. in Δ, and local decay holds

$$\| \langle x \rangle^{-\sigma} e^{-iHt} g_\Delta(H)\phi_0 \|_{L^2} = O(t^{-r})$$

Here $\langle x \rangle^{-\sigma}$ is weight coming from conditions (H).

c) $$e^{-iHt} g_\Delta(H)\phi_0 = (1+O(\varepsilon))\left(e^{-i\omega_* t} a(0)\psi_0 + \right.$$
$$\left. + e^{-iH_0 t}\phi_d(0)\right) + R(t)$$

where $a(0)\psi_0 = P_0 \phi_0$ the projection of initial data on ψ_0.

$$\phi_d(0) = P_c^{\#} \phi_0$$

$P_c^\#$ modified projection on cont. spec. (H_0).

$$\| \langle x \rangle^{-\sigma} R(t) \|_{L^2} \leq C \varepsilon \| W_\epsilon \| \quad \forall \; t \geq 0$$

d) If $\eta < 1$ and $\epsilon \to 0$, $R(t) = O(\varepsilon^2 \Gamma^{\eta - 1})$
and as $t\Gamma \to \infty$, $R(t) = O(\Gamma^{-1} t^{-\eta - 1})$

If $\eta > 1$, $R(t) = O(\varepsilon^2 t^{-\eta + 1})$.

$$-i\omega_* = -is_0 - \Gamma$$

s_0 solves:

$$s_0 + \omega + \varepsilon^2 \, \mathrm{Im} \{ F(\varepsilon, is_0) \} = 0$$

$$\Gamma = \epsilon^2 \, \mathrm{Re} \{ F(\varepsilon, is_0) \}$$

Condition (H)

(H1) H_0 is s.a. on $\mathcal{D}$ -dense in $L^2(\mathbb{R}^n)$

(H2) λ_0 is a simple emb. e.v. of H_0, with e.function ψ_0, $\|\psi_0\|_{L^2} = 1$.

(H3) $\exists$ interval $\Delta \ni \lambda_0$, with no other e.values.

(H4) For some $\sigma > 0$, local decay holds

$$\| \langle x \rangle^{-\sigma} e^{-iH_0 t} P_c^{\#} f \|_{L^2} \le c\, t^{-1-\eta} \| \langle x \rangle^{\sigma} f \|_{L^2}$$

$$\langle x \rangle^2 \equiv 1 + |x|^2,$$

$\eta > 0$.

$$P_c^{\#} = I - P_0 - P_1 = g_{\tilde{\Delta}}(H_0)$$

$\tilde{\Delta} \supset \Delta$.

(H5) By choosing c real,

$\langle x \rangle^{\sigma} (H_0 + c)^{-1} \langle x \rangle^{-\sigma}$ can be made suff. small.

Condition (W)

(W1) W_ϵ- symmetric and H_0 bounded with bound less than 1.

(W2)

$$|||W||| = \|\langle x\rangle^{2\sigma} W g_\Delta(H_0)\| + \|\langle x\rangle^{\sigma} W g_\Delta \langle x\rangle^{\sigma}\|$$

$$+ \|\langle x\rangle^{\sigma} W (H_0+c)^{-1}\langle x\rangle^{-\sigma}\| < \infty$$

and $\|\langle x\rangle^{\sigma} W (H_0+c)^{-1}\langle x\rangle^{\sigma}\| < \infty$

(W3) Resonance Condition

For $\eta > 1$, $\Gamma \neq 0$ and

$$\Gamma(\lambda,\varepsilon) = \pi\epsilon^2 (W_\epsilon \psi_0, \delta(H_0-\lambda)(I-P_0) W_\epsilon \psi_0)$$

For $\eta < 1$, $\Gamma \geq C\varepsilon^n$, $n \geq 2$,

$$\eta > \frac{n-2}{n}$$

Method

$$e^{-iHt}\phi_0 = a(t)\psi_0 + \tilde{\phi}(t) \qquad (*)$$

$$(\psi_0, \tilde{\phi})_{L^2} = 0 \quad \forall t$$

$$I = P_0 + P_1 + P_c^{\#}$$

Apply P_0: (scalar product with ψ_0 of $(\tilde{*})$)

$$i\partial_t \tilde{\phi} = H_0\tilde{\Phi} + \varepsilon W_\epsilon \tilde{\phi} - (i\partial_t a - \lambda_0 a)\psi_0 \qquad (\tilde{*})$$
$$+ a\epsilon W \psi_0$$

to get:

$$i\partial_t a = (\lambda_0 + (\psi_0, \varepsilon W_\epsilon \psi_0))a + (\psi_0, \varepsilon W_\epsilon P_1 \tilde{\Phi})$$
$$+ (\psi_0, \epsilon W_\epsilon \phi_d)$$

$$\phi_d \equiv P_c^{\#}\tilde{\phi}$$

and similar eq. for $i\partial_t \phi_d = H_0 \phi_d + \cdots$

$*$

P_1 - trick

Eliminate $P_1 \tilde{\phi}$ term, using $g_\Delta(H)\phi = \phi$ to get

$$i\partial_t \phi_d = H_0 \phi_d + a P_c^{\#} \epsilon W_\epsilon \tilde{g}_\Delta(H)\psi_0$$
$$+ P_c^{\#} \epsilon W_\epsilon \tilde{g}_\Delta \phi_d$$

$$i\partial_t a = (\lambda_0 + O(\varepsilon))a + (\psi_0, \varepsilon W_\epsilon \tilde{g}_\Delta \phi_d)$$

Iterate (M. Weinstein + A.S.)

Laplace Transform (O. Costin + A.S.)

Iteration works for time dep. potentials, nonlinear dispersive equations, H-Fock with many bound states ...

Solving for $\hat{a}(p)$ we have:

$$i p \hat{a} = \omega \hat{a} + i a(0) - i \epsilon^2 F(\epsilon, p) \hat{a}$$

$$F(\epsilon, p) = \left(\tilde{\tilde{\psi}}_0, \left[I + \frac{i}{p + iH_0} P_\epsilon^{\#} W \tilde{g}_\Delta \right]^{-1} \frac{-i}{p + iH_0} \tilde{\psi}_0 \right)$$

$$\tilde{\psi}_0 \equiv W_\epsilon \tilde{g}_\Delta \psi_0 ,$$

so,

$$\hat{a}(p) = \frac{i a(0)}{ip - \omega + i\epsilon^2 F(\epsilon, p)}$$

$F(\epsilon, p)$ is η regular.

Def. of Γ

We are interested in $p = is$, $s \in \mathbb{R}$.

Let: $F \equiv F_1 + i F_2$

Then one can prove that (ϵ small)

$$s + \omega + \epsilon^2 F_2(\epsilon, is) = 0$$

has at least one root, s_0.

$$\Gamma = \epsilon^2 F_1(\epsilon, is_0)$$

note : $is_0 = -i\omega +$ corrections

Analytic Case

In this case, we get a true pole, that is a resonance by the usual def.. Furthermore, we have

Theorem 2

with an appropriate exponential cutoff function $g_\Delta(x)$, the remainder term decays as a stretched exponential times an asymptotic series.

Recent Results

* Gamow state $\Rightarrow$ resonance
* WKB used for large W and inf. # of embedded e. values.

Some Open Problems

1) The case of nonrel. QED:
In this case $\langle x \rangle^{\sigma} W$ is not bounded.

2) Many e.values in the interval Δ:
the semiclassical limit.
In this case $a(t) \to \vec{a}(t)$.

3) The membrane problem
Elastic string/membrane coupled to a dispersive (acustic waves, say) field.
In this case $\vec{a}(t)$ is inf. dimensional.