Critical Phenomena in Random Matrix Models

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Introduction

We consider the unitary ensemble of random matrices,

$$d\mu_N(M) = Z_N^{-1} \exp\left(-\frac{N}{T}\text{Tr } V(M)\right) dM,$$

where

$$Z_N = \int_{\mathcal{H}_N} \exp\left(-\frac{N}{T}\text{Tr } V(M)\right) dM,$$

on the space $\mathcal{H}_N$ of Hermitian $N \times N$ matrices $M = \left(M_{ij}\right)_{1 \le i,j \le N}$, where $V(x)$ is a polynomial,

$$V(x) = v_p x^p + v_{p-1} x^{p-1} + \dots ,$$

of an even degree p with $v_p > 0$.

The ensemble of eigenvalues $\lambda = \{\lambda_j, \ j = 1, \dots, N\}$ of M is given then by the formula

$$d\mu_N(\lambda) = \widetilde{Z}_N^{-1} \exp\left(-NH_N(\lambda)\right) d\lambda,$$

where

$$\widetilde{Z}_N = \int_{\Lambda_N} \exp\left(-NH_N(\lambda)\right) d\lambda,$$

where Λ_N is the symmetrized $\mathbb{R}^N$,

$$\Lambda_N = \mathbb{R}^N / S(N),$$

and

$$H_N(\lambda) = -\frac{2}{N} \sum_{1 \le j < k \le N} \log|\lambda_j - \lambda_k| + \frac{1}{T} \sum_{j=1}^{N} V(\lambda_j)\,.$$

Let $d\nu_N(x) = \rho_N(x)dx$ be the distribution of the eigenvalues on the line, so that for any test function $\phi(x) \in C_0^\infty$,

$$\int_{\Lambda_N} \left[\frac{1}{N} \sum_{j=1}^{N} \phi(\lambda_j) \right] d\mu_N(\lambda) = \int_{-\infty}^{\infty} \phi(x)\, d\nu_N(x)\,.$$

As $N \to \infty$, there exists a weak limit of $d\nu_N(x)$,

$$d\nu_\infty(x) = \lim_{N\to\infty} d\nu_N(x)\,.$$

To determine the limit consider the energy functional on the space of probability measures on the line,

$$\begin{aligned} I(d\nu(x)) = &-\frac{\beta}{2} \iint_{\mathbb{R}^2} \log|x-y| d\nu(x)\, d\nu(y) \\ &+ \frac{1}{T} \int_{\mathbb{R}} V(x) d\nu(y)\,, \end{aligned}$$

where $\beta = 2$.

Then

$$H_N(\lambda) = NI(d\nu(x;\lambda))\,,$$

where

$$d\nu(x;\lambda) = \frac{1}{N}\sum_{j=1}^{N} \delta(x-\lambda_j)\,dx\,.$$

Hence,

$$d\mu_N(\lambda) = \widetilde{Z}_N^{-1} \exp\left(-N^2 I(d\nu(x;\lambda))\right) d\lambda\,.$$

Because of the factor N^2 in the exponent, one can expect that as $N \to \infty$, the measures $d\mu_N(\lambda)$ are localized in a shrinking vicinity of an *equilibrium measure* $d\nu_{\text{eq}}(x)$, which minimizes the functional $I(d\nu(x))$, and therefore, one expects the limit to exist with $d\nu_\infty(x) = d\nu_{\text{eq}}(x)$.

A rigorous proof of the existence of the limit $\lim_{N\to\infty} d\nu_N(x) = d\nu_{\text{eq}}(x)$, was given by A. Boutet de Monvel, L. Pastur, and M. Shcherbina, and by K. Johansson.

A. Boutet de Monvel, L. Pastur, and M. Shcherbina, On the statistical mechanics approach in the random matrix theory: Integrated density of states, J. Statist. Phys. **79** (1995), 585–611.

K. Johansson, On fluctuations of eigenvalues of random hermitian matrices, Duke Math. J. **91** (1988), 151–204.

For the existence and uniqueness of the equilibrium measure and its analytic properties see also

E. Saff and V. Totik, Logarithmic potentials and external fields, Springer-Verlag, New York, 1997.

P. Deift, T. Kriecherbauer, K. T-R. McLaughlin, New results on the equilibrium measure for logarithmic potentials in the presence of an external field, J. Appr. Theory **95** (1998) 399-475.

Properties of the equilibrium measure

- $d\nu_{\mathrm{eq}}(x)$ is supported by a finite number of segments $[a_j, b_j]$, $j = 1, \dots, q$, and it is absolutely continuous with respect to the Lebesgue measure, $d\nu_{\mathrm{eq}}(x) = \rho(x)dx$,

- the density function $\rho(x)$ is of the form

$$\rho(x) = \frac{1}{2\pi i} h(x) R_+^{1/2}(x),$$
$$R(x) = \prod_{j=1}^{q} (x - a_j)(x - b_j),$$

where $h(x)$ is a polynomial of the degree, $\deg h = p - q - 1$, and $R_+^{1/2}(x)$ means the value on the upper cut of the principal sheet of the function $R^{1/2}(z)$ with cuts on J,

$$J = \cup_{j=1}^{q} [a_j, b_j] \, .$$

The equilibrium measure is uniquely determined by the Euler-Lagrange conditions: for some real constant l,

$$2\int_{\mathbb{R}} \log|x-s|\, d\nu_{\text{eq}}(s) - V(x) = l, \qquad \text{for} \quad x \in \cup_{j=1}^{q}[a_j, b_j]\,;$$

$$2\int_{\mathbb{R}} \log|x-s|\, d\nu_{\text{eq}}(s) - V(x) \le l, \qquad \text{for} \quad x \in \mathbb{R} \setminus \cup_{j=1}^{q}[a_j, b_j]\,.$$

See

P. Deift, T. Kriecherbauer, K. T-R. McLaughlin, S. Venakides, and X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, *Commun. Pure Appl. Math.*, **52** (1999) 1335-1425

The Euler-Lagrange conditions imply that

$$\omega(z) = \frac{V'(z)}{2} - \frac{h(z)R^{1/2}(z)}{2}, \qquad (*)$$

where

$$\omega(z) \equiv \int_J \frac{\rho(x)\,dx}{z-x} = z^{-1} + O(z^{-2}), \quad z \to \infty\,,$$

and

$$\int_{b_j}^{a_{j+1}} \frac{h(x)R^{1/2}(x)}{2}\,dx = 0\,, \quad j = 1, \dots, q-1\,, \qquad (**)$$

which shows that $h(x)$ has at least one zero on each interval $b_j < x < a_{j+1}$; $j = 1, \dots, q-1$.

From (*) we obtain that

$$\begin{aligned} V'(z) &= \text{Pol}\left[h(z)R^{1/2}(z)\right], \\ \underset{z=\infty}{\text{Res}}\left[h(z)R^{1/2}(z)\right] &= -2, \end{aligned} \qquad (***)$$

and

$$h(z) = \text{Pol}\left[\frac{V'(z)}{R^{1/2}(z)}\right],$$

where Pol $[f(z)]$ is the polynomial part of $f(z)$ at $z = \infty$. The latter equation expresses $h(z)$ in terms of $V(z)$ and the end-points, $a_1, b_1, \ldots,$ a_q, b_q. The end-points can be further found from (***), which gives $q + 1$ equation on $a_1, \ldots, b_q$, and from (**), which gives the remaining $q - 1$ equation.

The equilibrium measure $d\nu_{\text{eq}}(x)$ is called *regular* (otherwise *singular*), see [DKMVZ], if

$$h(x) \neq 0 \quad \text{for} \quad x \in \cup_{j=1}^{q}[a_j, b_j]$$

and

$$2\int \log|x-s|\, d\nu_{\text{eq}}(s) - V(x) < l, \qquad \text{for} \quad x \in \mathbb{R} \setminus \cup_{j=1}^{q}[a_j, b_j]\,.$$

The polynomial $V(x)$ is called *critical* if the corresponding equilibrium mesaure $d\nu_{\text{eq}}(x)$ is singular. If $V(x)$ is a critical polynomial then the set S of its *singular points* consists of the points where either $h(x) = 0$, $x \in \cup_{j=1}^{q}[a_j, b_j]$, or

$$2\int \log|x-s|\, d\nu_{\text{eq}}(s) - V(x) = l, \qquad x \in \mathbb{R} \setminus \cup_{j=1}^{q}[a_j, b_j]\,.$$

Free Energy and Eigenvalue Correlation Functions

Free energy.

$$F_N = -\frac{T}{N^2} \ln Z_N,$$
$$Z_N = \int_{\mathcal{H}_N} \exp\left(-\frac{N}{T} \operatorname{Tr} V(M)\right) dM.$$

Proposition **1.** *There exists the limit,*

$$F = \lim_{N\to\infty} F_N\,.$$

Correlation functions. For any test function $\phi(x) \in C_0^\infty$, define

$$\phi[\lambda] = \sum_{j=1}^{N} \phi(\lambda_j)\,, \qquad \lambda = \{\lambda_1, \ldots, \lambda_N\}\,.$$

The m-point correlation measure $d\nu_{mN}(x_1, \ldots, x_m)$ is defined by the condition that for any m test functions $\phi_1(x), \ldots, \phi_m(x)$,

$$\int_{\Lambda_N} \left(\prod_{j=1}^{m} \phi_j[\lambda] \right) d\mu_N(\lambda)$$
$$= \int_{\mathbb{R}^m} \left(\prod_{j=1}^{m} \phi_j(x_j) \right) d\nu_{mN}(x_1, \ldots, x_m)\,.$$

In fact, $d\nu_N(x_1, \ldots, x_m)$ is Lebesgue absolutely continuous on the set $\{x_j \neq x_k\}$,

$$d\nu_N(x_1, \ldots, x_m) = K_{mN}(x_1, \ldots, x_m)\, dx_1 \ldots dx_m\,,$$

and $K_{mN}(x_1, \ldots, x_m)$ is the m-point correlation function.

Orthogonal Polynomials

Consider monic orthogonal polynomials $P_n(x) = x^n + \dots$,

$$\int_{-\infty}^{\infty} P_n(x)P_k(x)e^{-NV(x)}dx = h_n\delta_{nk},$$

Define

$$\psi_n(x) = \frac{1}{h_n^{1/2}}P_n(x)e^{-NV(x)/2}.$$

Recurrence relation:

$$x\psi_n(x) = \gamma_{n+1}\psi_{n+1}(x) + \beta_n\psi_n(x) + \gamma_n\psi_{n-1}(x).$$

Consider the matrix Q of the operator of multiplication by x, $f(x) \to xf(x)$ in the basis $\{\psi_n(x)\}$; Q is a symmetric tridiagonal matrix,

$$Q = \begin{pmatrix} \beta_0 & \gamma_1 & 0 & \dots \\ \gamma_1 & \beta_1 & \gamma_2 & \dots \\ 0 & \gamma_2 & \beta_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Discrete String Equations

$$\begin{cases} [V'(Q)]_{nn} = 0, \\ \gamma_n [V'(Q)]_{n,n-1} = \dfrac{n}{N}. \end{cases}$$

Lax Pair Equations

Define $\vec{\Psi}_n(x) = \begin{pmatrix} \psi_n(x) \\ \psi_{n-1}(x) \end{pmatrix}$.

Differential equation:

$$\vec{\Psi}'_n(x) = NA_n(x)\vec{\Psi}_n(x),$$

where

$$A_n(x) = \begin{pmatrix} -\frac{V'(x)}{2} - \gamma_n u_n(x) & \gamma_n v_n(x) \\ -\gamma_n v_{n-1}(x) & \frac{V'(x)}{2} + \gamma_n u_n(x) \end{pmatrix}$$

and

$$u_n(x) = [W(Q,x)]_{n,n-1},$$
$$v_n(x) = [W(Q,x)]_{nn},$$

where

$$W(Q,x) = \frac{V'(Q) - V'(x)}{Q - x}.$$

Recurrence equation:

$$\vec{\Psi}_{n+1}(x) = U_n(x)\vec{\Psi}_n(x),$$

where

$$U_n(x) = \begin{pmatrix} \gamma_{n+1}^{-1}(x-\beta_n) & -\gamma_{n+1}^{-1}\gamma_n \\ 1 & 0 \end{pmatrix}$$

Correlation functions in terms of orthogonal polynomials

Formula of Dyson

$$K_{mN}(x_1,\dots,x_m) = \det\left(Q_N(x_k,x_l)\right)_{k,l=1}^{m},$$

where

$$Q_N(x,y) = \sum_{n=0}^{N-1} \psi_n(x)\psi_n(y)$$

Christoffel-Darboux Formula

$$\begin{aligned} Q_N(x,y) &= \sum_{n=0}^{N-1} \psi_n(x)\psi_n(y) \\ &= \gamma_N \frac{\psi_N(x)\psi_{N-1}(y) - \psi_{N-1}(x)\psi_N(y)}{x-y}. \end{aligned}$$

Theorem **2.** *If the equilibrium measure,*

$$d\nu_{\mathrm{eq}}(x; T, v_1, \dots, v_p),$$

is regular (in the sense of [DKMVZ]) then the free energy $F(T, v_1, \dots, v_p)$ *is analytic in* T, $v_1, \dots, v_p$ *(at a given point).*

We will be interested in the following problems:

- Singularity of F at critical points.
- Asymptotic expansion of $F_N - F$ as $N \to \infty$ both at regular and critical points.
- Double scaling limit of correlation functions at critical points.

We will consider the critical density

$$\rho_c(x) = \frac{1}{2\pi T_c}(x - 2c_1)^2\sqrt{4 - x^2},$$

where $T_c = 1 + 4c_1^2$ and

$$c_1 = \cos \pi\epsilon, \qquad -1 < \epsilon < 1\,.$$

(Many results are extended to the critical density of a more general form:

$$\rho_c(x) = h_0(x)(x - 2c_1)^2\sqrt{4 - x^2},$$

where $h_0(x) > 0$ for real x.) The critical polynomial is

$$V_c'(x) = \frac{1}{T_c}V(x),$$
$$V(x) = \frac{1}{4}x^4 - \frac{4}{3}c_1x^3 + c_2x^2 + 8c_1x,$$

where we denote

$$c_k = \cos k\pi\epsilon\,, \qquad s_k = \sin k\pi\epsilon.$$

Free energy near the critical point. To evaluate the singularity the free energy $F(T)$ at $T = T_c$, consider the function

$$
\begin{aligned}
F_1(T) &= T^2 \frac{d}{dT}\left(\frac{F(T)}{T}\right) \\
&= \lim_{N\to\infty} \frac{1}{Z_N(T)} \int_{\mathcal{H}_N} \frac{1}{N} \operatorname{Tr} V(M) \\
&\times \exp\left(-\frac{N}{T} \operatorname{Tr} V(M)\right) dM.
\end{aligned}
$$

It can be evaluated as

$$
F_1(T) = \frac{1}{2\pi i} \oint_C V(z)\omega(z;T)\, dz,
$$

where C is any contour with positive orientation around $J(T)$, the support of equilibrium measure and

$$
\omega(z;T) = \int_{J(T)} \frac{\rho(x;T)\, dx}{z - x}.
$$

Theorem **3.** *The free energy $F(T)$ can be analytically continued through $T = T_c$ both from below and from above T_c. In addition, $F(T)$ and its first two derivatives, $F'(T)$, $F''(T)$ are continuous at $T = T_c$, while the third derivative, $F'''(T)$ has a jump at $T = T_c$ (third order phase transition).*

(Extension of a result in

D. Gross and E. Witten, Possible third-order phase transition in the large-N lattice gauge theory, Phys. Rev. D **21** (1980) 446-453.)

Double Scaling Limit for Recurrence Coefficients

We start with the following ansatz, which reproduces the quasiperiodic behavior of the recurrence coefficients:

$$
\begin{aligned}
\frac{n}{N} &= 1 + N^{-2/3}t, \\
\gamma_n^2 &= 1 + N^{-1/3}u(t)\cos 2n\pi\epsilon \\
&\quad + N^{-2/3}\left[v_0(t) + v_1(t)\cos 2n\pi\epsilon \right.\\
&\quad \left. + v_2(t)\cos 4n\pi\epsilon\right] \\
&\quad + N^{-1}\left[w_0(t) + w_1(t)\cos 2n\pi\epsilon \right.\\
&\quad + w_2(t)\cos 4n\pi\epsilon + w_3(t)\cos 6n\pi\epsilon \\
&\quad \left. + w_4(t)\sin 4n\pi\epsilon\right], \\
\beta_n &= 0 + N^{-1/3}u(t)\cos(2n+1)\pi\epsilon \\
&\quad + N^{-2/3}\left[\tilde{v}_0(t) + \tilde{v}_1(t)\cos(2n+1)\pi\epsilon \right.\\
&\quad \left. + \tilde{v}_2(t)\cos(4n+2)\pi\epsilon\right] \\
&\quad + N^{-1}\left[\tilde{w}_0(t) + \tilde{w}_1(t)\cos(2n+1)\pi\epsilon \right.\\
&\quad + \tilde{w}_2(t)\cos(4n+2)\pi\epsilon + \tilde{w}_3(t)\cos(6n+3)\pi\epsilon \\
&\quad \left. + \tilde{w}_4(t)\sin(4n+2)\pi\epsilon\right],
\end{aligned}
$$

where $u(t)$, $v_0(t), \dots, \tilde{w}_4(t)$ are unknown functions.

We substitute the ansatz into string equations.

Order $N^{-1/3}$. Our ansatz is automatically satisfied at this order.

Order $N^{-2/3}$. We obtain that

$$v_0 = -\frac{c_1^2}{4s_1^2}u^2 + \frac{1+c_1^2}{4s_1^4}tT_c,$$
$$\widetilde{v}_0 = -\frac{c_1}{4s_1^2}u^2 + tT_c\frac{c_1}{2s_1^4},$$

and

$$\widetilde{v}_1 - v_1 = \frac{1}{2}u',$$
$$v_2 = \frac{u^2}{4s_1^2},$$
$$\widetilde{v}_2 = \frac{c_1 u^2}{4s_1^2}.$$

Order N^{-1}. We obtain $w_0, \ldots, \widetilde{w}_4$ and a nonlinear equation on $u(t)$,

$$\boxed{2s_1^2 u'' = u^3 + \frac{T_c}{s_1^2} tu,}$$

which is the Painlevé II equation. When $\epsilon = 1/2$ it reduces to $2u'' = u^3 + tu$. The function $u(t)$ behaves as

$$u(t) \underset{t\to-\infty}{\sim} \frac{1}{s_1}\sqrt{-T_c t}\,, \qquad u(t) \underset{t\to+\infty}{\sim} \mathrm{Ai}\,(\kappa\, t),$$

$\kappa = \left(\frac{T_c}{2s_1^4}\right)^{1/3}$. (The Hastings-McLeod solution of Painlevé II).

Scaled Differential Equations at the Critical Point

To derive a scaled system at the critical point $x = 2c_1$ we set

$$x = 2c_1 + yN^{-1/3},$$

and

$$\psi_n(x) = \cos\left(n + \frac{1}{2}\right)\pi\epsilon\, f(y) - \sin\left(n + \frac{1}{2}\right)\pi\epsilon\, g(y),$$

$$\psi_{n-1}(x) = \cos\left(n - \frac{1}{2}\right)\pi\epsilon\, f(y) - \sin\left(n - \frac{1}{2}\right)\pi\epsilon\, g(y),$$

The scaled system is

$$\frac{T_c}{s_1}\frac{d}{dy}\begin{pmatrix} f \\ g \end{pmatrix} =$$

$$\begin{pmatrix} s_1 u' & \left(y^2 + \frac{u^2}{2} + \frac{Tt}{2s_1^2}\right) + yu \\ -\left(y^2 + \frac{u^2}{2} + \frac{Tt}{2s_1^2}\right) + yu & -s_1 u' \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

the differential ψ-equation for Painlevé II equation.

Universal Kernel for Correlation Functions

We scale:

$$\frac{n}{N} = 1 + N^{-2/3} \left(\frac{2s_1^4}{T_c} \right)^{1/3} \tilde{t},$$

$$x = 2c_1 + N^{-1/3} \left(\frac{4T_c}{s_1} \right)^{1/3} \tilde{y}\,.$$

The Dyson integral kernel for the double scaling limit correlation functions is then:

$$\boxed{K(\tilde{y}_1, \tilde{y}_2) = \frac{f(\tilde{y}_1)g(\tilde{y}_2) - g(\tilde{y}_1)f(\tilde{y}_2)}{\tilde{y}_1 - \tilde{y}_2}\,.}$$

Nonlinear Hierarchy

For $m = 1, 2, \ldots$, we consider the model critical density

$$\rho(x) = \frac{1}{2\pi T_c}(x - 2c_1)^{2m}\sqrt{4 - x^2},$$

where

$$T_c = \frac{1}{2\pi}\int_{-2}^{2}(x - 2c_1)^{2m}\sqrt{4 - x^2}\,dx.$$

The corresponding polynomial $V(x)$ is such that

$$V'(x) = \frac{1}{T_c}\,\mathrm{Pol}\left[(x - 2c_1)^{2m}\sqrt{4 - x^2}\right],$$

In the double scaling limit we define variables K, t and y as

$$K = N^{-1/(2m+1)}, \qquad \frac{n}{N} = 1 + K^{2m} s_1 t,$$
$$x = 2c_1 + 2Ky.$$

Our ansatz for the orthogonal polynomials is the following:

$$\begin{aligned}
&\psi(n, x) \\
&= \cos(n + 1/2)\pi\epsilon\, f(t, y) - \sin(n + 1/2)\pi\epsilon\, g(t, y) \\
&+ K\,[\cos(n + 1/2)\pi\epsilon\, f_1(t, y) - \sin(n + 1/2)\pi\epsilon\, g_1(t, y) \\
&+ \cos 3(n + 1/2)\pi\epsilon\, \widetilde{f}(t, y) - \sin 3(n + 1/2)\pi\epsilon\, \widetilde{g}(t, y)\Big] \\
&+ O(K^2),
\end{aligned}$$

and for the recurrence coefficients,

$$\gamma_n = 1 + Ku(t)\cos 2n\pi\epsilon + O(K^2),$$
$$\beta_n = 2Ku(t)\cos(2n + 1)\pi\epsilon + O(K^2)$$

When we substitute the ansatz into the 3-term recurrence relation, we obtain the equation

$$\partial_t \begin{pmatrix} f(t,y) \\ g(t,y) \end{pmatrix} = L \begin{pmatrix} f(t,y) \\ g(t,y) \end{pmatrix},$$

$$L = \begin{pmatrix} 0 & y + u(t) \\ -y + u(t) & 0 \end{pmatrix}$$

We would like to derive a differential equation in y,

$$\partial_y \begin{pmatrix} f(t,y) \\ g(t,y) \end{pmatrix} = D(t,y) \begin{pmatrix} f(t,y) \\ g(t,y) \end{pmatrix}.$$

We are looking for $D(t,y)$ in the form

$$D(t,y) = \begin{pmatrix} -A(t,y) & yB(t,y) + C(t,y) \\ yB(t,y) - C(t,y) & A(t,y) \end{pmatrix},$$

where A, B and C are even polynomials in y of the following degrees:

$$\deg A = 2m-2, \quad \deg B = 2m-2,$$
$$\deg C = 2m.$$

We will assume that C is a monic polynomial, so that $C = y^{2m} + \ldots$. The general case can be reduced to this one by the change of variables, $t = \kappa\tilde{t}$, $y = \frac{\tilde{y}}{\kappa}$, $u(t) = \frac{\tilde{u}(\tilde{t})}{\kappa}$, which preserves the structure of the operator L.

The consistency condition of the differential equations in t and y,

$$[D, L] = \partial_y L - \partial_t D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \partial_t D,$$

implies that

$$\partial_t B = 2A\,, \quad \partial_t C = 1 + 2uA\,,$$
$$\partial_t A = -2y^2 B + 2uC\,.$$

We would like to solve these equations in polynomials of the degree

$$\deg A = 2m - 2, \quad \deg B = 2m - 2,$$
$$\deg C = 2m.$$

Define recursively functions $A_m(t,y)$, $B_m(t,y)$, $C_m(t,y)$ by the equations

$$
\begin{aligned}
C_{m+1} &= y^2 C_m + f_m(u)\,,\\
B_{m+1} &= y^2 B_m + R_m(u)\,,\\
A_{m+1} &= y^2 A_m + \frac{1}{2}\,\partial_t R_m(u)\,,
\end{aligned}
$$

where $R_m(u)$, $f_m(u)$ solve the recursive equations

$$
\begin{aligned}
R_{m+1}(u) &= u f_m(u) - \frac{1}{4}\,\partial_{tt} R_m(u),\\
\partial_t f_m(u) &= u \partial_t R_m(u), \qquad f_m(0) = 0,
\end{aligned}
$$

with the initial data

$$
\begin{aligned}
&A_0 = B_0 = 0, \quad C_0 = 1, \quad R_0(u) = u,\\
&f_0(u) = \frac{u^2}{2}\,.
\end{aligned}
$$

We find that

$$R_1(u) = \frac{1}{2}u^3 - \frac{1}{4}u'',$$

$$f_1(u) = \frac{3}{8}u^4 - \frac{1}{4}uu'' + \frac{1}{8}u'^2;$$

$$R_2(u) = \frac{3}{8}u^5 - \frac{5}{8}u^2u'' - \frac{5}{8}uu'^2 + \frac{1}{16}u^{(4)},$$

$$f_2(u) = \frac{5}{16}u^6 - \frac{5}{8}u^3u'' - \frac{5}{16}u^2u'^2 + \frac{1}{16}uu^{(4)} - \frac{1}{16}u'u''' + \frac{1}{32}u''^2;$$

$$R_3(u) = \frac{5}{16}u^7 - \frac{35}{32}u^4u'' - \frac{35}{16}u^3u'^2 + \frac{7}{32}u^2u^{(4)} + \frac{7}{8}uu'u''' + \frac{21}{32}uu''^2 + \frac{35}{32}u'^2u'' - \frac{1}{64}u^{(6)},$$

and so on,

$$R_m(u) = \frac{(2m)!}{2^{2m}(m!)^2}u^{2m+1} + \ldots + \frac{(-1)^m}{2^{2m}}u^{(2m)}.$$

In addition,

$$A_1 = \frac{1}{2}u', \qquad B_1 = u, \qquad C_1 = y^2 + \frac{1}{2}u^2;$$

$$A_2 = \frac{1}{2}u'y^2 + \frac{3}{4}u^2u' - \frac{1}{8}u''',$$

$$B_2 = uy^2 + \frac{1}{2}u^3 - \frac{1}{4}u'',$$

$$C_2 = y^4 + \frac{1}{2}u^2y^2 + \frac{3}{8}u^4 - \frac{1}{4}uu'' + \frac{1}{8}(u')^2;$$

and so on.

Theorem **4.** *Define*

$$D_m(t,y) = \begin{pmatrix} -A_m(t,y) & yB_m(t,y) + C_m(t,y) \\ yB_m(t,y) - C(t,y) & +A_m(t,y) \end{pmatrix}$$

Then if $u(t)$ *is a solution of the equation*

$$R_m(u) + tu = 0\,.$$

(the Painleve II hierarchy), then the matrix

$$D(t,y) = \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} + D_m(t,y)$$

is a solution to the equation

$$[D,L] = \partial_y L - \partial_t D. \qquad (\dagger)$$

More generally, if $t_1, \ldots, t_m$ *are arbitrary constants and* $u(t)$ *is a solution of the equation*

$$\sum_{k=1}^{m} t_k R_k(u) + tu = 0,$$

then the matrix

$$D(t,y) = \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} + \sum_{k=1}^{m} t_k D_k(t,y) \qquad (\dagger\dagger)$$

is a solution to ($\dagger$).

The meaning of the constants $t_1, \ldots, t_m$ is the following. Observe that the differential equation in y describes the double scaling limit for a critical polynomial of degeneracy $2m$. In this case the space of transversal fluctuations to the manifold of critical polynomials has dimension m. The variables $t_1, \ldots, t_m$ serve as coordinates in the space of transversal fluctuations, and the above general solution gives the matrix describing the double scaling limit of the recurrence coefficients in the direction $\tau = (t_1, \ldots, t_m)$.

Some References for the Double Scaling Limit

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