The RH method as a non-commutative FL method

Basic example: Painlevé II

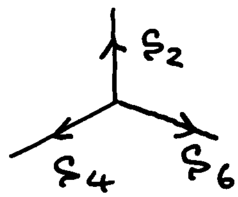
$$u_{xx} - xu - 2u^3 = 0$$

$$\uparrow$$

$$u_{xx} - xu = 0 \qquad \text{Airy}$$

Linear case: $\underline{u_{xx} = xu}$

$$u(x) = \frac{i}{\pi} \int e^{-2i\theta(\lambda, x)} \, d\lambda$$



$$\theta(\lambda) = \frac{4}{3}\lambda^3 + x\lambda , \qquad s_2 + s_4 + s_6 = 0$$

$$\equiv \frac{i}{\pi} \sum_{k=2,4,6} s_k \int_{\Gamma_k} e^{-2i\theta(\lambda, x)} \, d\lambda$$

$$\equiv \frac{i}{\pi} \int_{\Gamma} e^{-2i\theta(\lambda)} \, d\lambda , \qquad \Gamma_k : \arg\lambda = \frac{2k-1}{6}\pi$$

Observe:

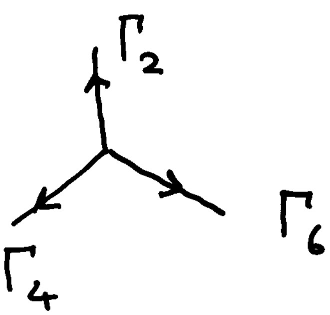
$$u(x) = 2 \lim_{\lambda \to \infty} \lambda Y_{12}(\lambda)$$

1. $Y(\lambda) \in H(\mathbb{C} \setminus \Gamma)$

2. $Y_+(\lambda) = Y_-(\lambda)\, G_L(\lambda) \;, \quad \lambda \in \Gamma$

3. $Y(\lambda) \longrightarrow I \;, \quad \lambda \longrightarrow \infty$

Γ :

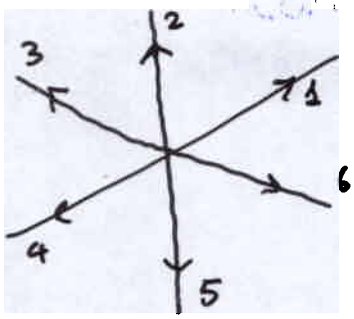


$$G_L(\lambda) = \begin{pmatrix} 1 & s_k e^{-2i\theta(\lambda)} \\ 0 & 1 \end{pmatrix} \quad \lambda \in \Gamma_k$$

$$\boxed{1\text{-}3 \implies Y(\lambda) \text{ uniquely !}}$$

Nonlinear case. ($\equiv$ non-abelian)

Γ :



$$\{s_k, \Gamma_k\}_{k=2,4,6} \longrightarrow \{s_k, \Gamma_k\}_{k=1,2,3,4,5,6}$$

$$G_L(\lambda, x) \longrightarrow G_{NL}(\lambda, x):$$

$$\begin{pmatrix} 1 & s_k e^{-2i\theta(\lambda)} \\ 0 & 1 \end{pmatrix} \quad k = 2, 4, 6$$

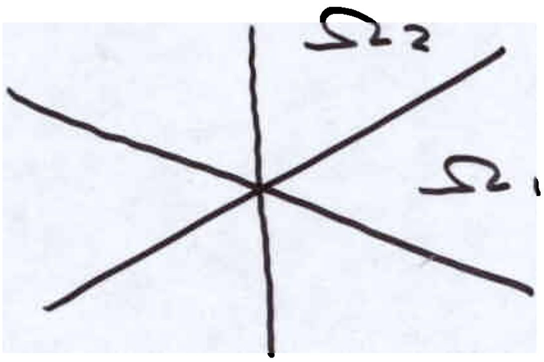
$$\begin{pmatrix} 1 & 0 \\ s_k e^{2i\theta(\lambda)} & 1 \end{pmatrix} \quad k = 1, 3, 5$$

$$s_{k+3} = -s_k \qquad \text{"}\Longleftrightarrow\text{"} \quad S_1 S_2 \cdots S_6 = I$$

$$s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0$$

$$Y_\pm(\lambda):\quad Y\big|_{\Omega_k} \in H(\Omega_k)\cap C(\bar\Omega_k)$$

Ω_k:



Theorem. (Novokshenov & I (88); Fokas & Zhou (94)

(?) also follows from general results of Miwa (81)
and Malgrange (81)
and from general results of
Mason, Singer & Woodhouse (90th)
and
Palmer (00)) (Balibrouch, Kapaev & I
(01))

$$\{s\} \longmapsto \omega_s = \{x_j\}_{j=1}^{\infty}, \quad x_j \to \infty$$

$Y(\lambda, x)$:

$\boxed{1.}$ $\left(Y\big|_{\Omega_k}\right)(\lambda, x)$ - holomorphic in

$$\overline{\Omega}_k \times (\mathbb{C} \setminus \omega_s)$$

and meromorphic along $\overline{\Omega}_k \times \omega_s$

$\boxed{2.}$ $Y_+(\lambda, x) = Y_-(\lambda, x)\, G_k(\lambda, x)$

$$\lambda \in \Gamma, \quad x \notin \omega_s$$

$\boxed{3.}$ $Y(\lambda, x) \cong I + \sum_{j=1}^{\infty} \frac{m_j(x)}{\lambda^j}, \quad \lambda \to \infty$

$$x \in K \subset \mathbb{C} \setminus \omega_s$$

diff. w.r.t. x, λ

$m_j(x)$ - meromorphic, $(m_j) = \omega_s$

4. $\boxed{u_{xx} = xu + 2u^3}$ $(*)$

$$u(x) = 2\lim_{\lambda\to\infty} (\lambda Y_{12}(\lambda, x))$$

$$= 2(m_1(x))_{12}$$

5. The map

$$\{ s : s_{k+3} = -s_k ,\ s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0 \}$$

$$\longmapsto \{ \text{solutions of } (*) \}$$

is a bijection

$$\Downarrow$$

6. Each solution of $(*)$ is a meromorphic function. (Painlevé property)

$(*)$ - Painlevé II equation.

The Γ - function

↑

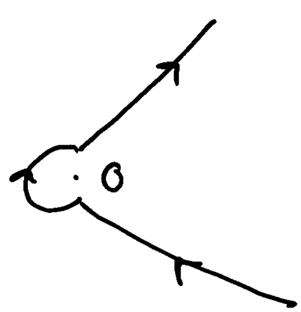
(A. Kitaev, Acta Applicadae Mathematical
64: 1-32, 2000
Special Functions of the Isomonodromy Type)

(A. Kapaev, A. Kitaev, A.I. 1988)

$$\gamma(x) := \frac{1}{2\pi i}\left(1 - e^{2\pi i x}\right)\Gamma(x+1)$$

$$= \frac{1}{2\pi i}\int_C e^{-\lambda}\lambda^{x}\, d\lambda$$

C :



$$\gamma(x) = - \lim_{\lambda \to \infty} \lambda Y_{12}(\lambda)$$

- $Y(\lambda) \in H(\mathbb{C} \setminus C)$

- $Y_+(\lambda) = Y_-(\lambda)\, G(\lambda; x)$

- $Y(\lambda) \longmapsto I, \quad \lambda \to \infty$

$$G(\lambda) = \begin{pmatrix} 1 & e^{-\lambda} \lambda^{x} \\ 0 & 1 \end{pmatrix}$$

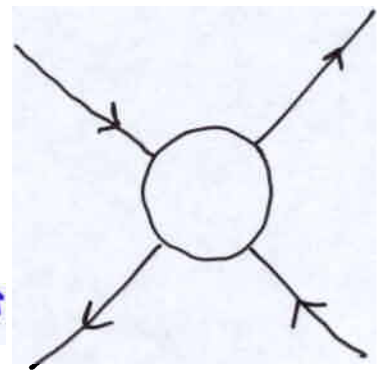
$$\Psi(\lambda) := Y(\lambda)\, e^{-\frac{\lambda}{2}\sigma_3}\, \lambda^{\frac{x}{2}\sigma_3}$$

$$\downarrow$$

$$\left\{\begin{array}{l} \dfrac{\partial \Psi}{\partial \lambda} = \underbrace{\left[-\dfrac{\sigma_3}{2} + \dfrac{1}{\lambda}\begin{pmatrix} x/2 & -\gamma \\ 0 & -x/2 \end{pmatrix} \right]}_{\equiv A(\lambda, x)} \Psi \\[4em] \Psi(\lambda, x+1) = \underbrace{\left[\sqrt{\lambda}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \dfrac{1}{\sqrt{\lambda}}\begin{pmatrix} 0 & \gamma \\ 0 & 1 \end{pmatrix} \right]}_{\equiv B(\lambda, x)} \Psi \end{array}\right.$$

$$\boxed{\begin{array}{c} \dfrac{\partial B}{\partial \lambda} = A(\lambda, x+1)\, B(\lambda, x) - B(\lambda, x)\, A(\lambda, x) \\[1em] \Updownarrow \\[1em] \gamma(x+1) = (x+1)\,\gamma(x) \end{array}}$$

"Non-abelian" Γ-function:



$$\begin{pmatrix} 1 & 0 \\ e^{-\lambda}\lambda^{+x} & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -e^{\lambda}\lambda^{-x} \\ 0 & 1 \end{pmatrix}$$

The ζ - function.

$$\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = -\frac{1}{s} + \frac{1}{s-1} + \int_0^1 \omega\left(\frac{1}{t}\right) t^{\frac{s}{2}-3/2}\,dt + \int_1^\infty \omega(t)\, t^{s/2-1}\,dt$$

$$\boxed{\mathring{\zeta}(s) := \int_0^1 \omega\left(\frac{1}{\lambda}\right)\lambda^{\frac{s}{2}-3/2}\,d\lambda + \int_1^\infty \omega(\lambda)\,\lambda^{\frac{s}{2}-1}\,d\lambda}$$

$$\omega(\lambda) = \sum_{n=1}^{\infty} e^{-n^2\pi\lambda} \equiv \frac{1}{2}\left[\theta_3(0;i\lambda)-1\right]$$

$$\overset{\circ}{\xi}(s) = -\lim_{\lambda \to \infty} \lambda\, Y_{12}(\lambda)$$

$G(\lambda)$:

$$\begin{pmatrix} 1 & 2\pi i\, \omega(\lambda)\, \lambda^{\frac{s}{2}-1} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2\pi i\, \omega(1/\lambda)\, \lambda^{s/2 - 3/2} \\ 0 & 1 \end{pmatrix}$$

- $Y(\lambda) \in H(\mathbb{C} \setminus \mathbb{R}_+)$
- $Y_+(\lambda) = Y_-(\lambda)\, G(\lambda)$
- $Y(\lambda) \longrightarrow I \qquad \lambda \to \infty$

(3.)

$$\Psi(\lambda, s) := Y(\lambda, s)\,\lambda^{s/4\,\sigma_3}$$

$$\begin{cases} \Psi(\lambda, s+2) = \left[\sqrt{\lambda}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{\sqrt{\lambda}}\begin{pmatrix} 0 & -\mathring{\zeta}(s) \\ 0 & 1 \end{pmatrix}\right]\Psi(\lambda, s) \\ \sigma_3\,\Psi\left(\frac{1}{\lambda}, 5-s\right)\sigma_3 = \begin{pmatrix} 1 & -\mathring{\zeta}(3-s) \\ 0 & 1 \end{pmatrix}\Psi(\lambda, s)\,\lambda^{-5/4\,\sigma_3} \end{cases}$$

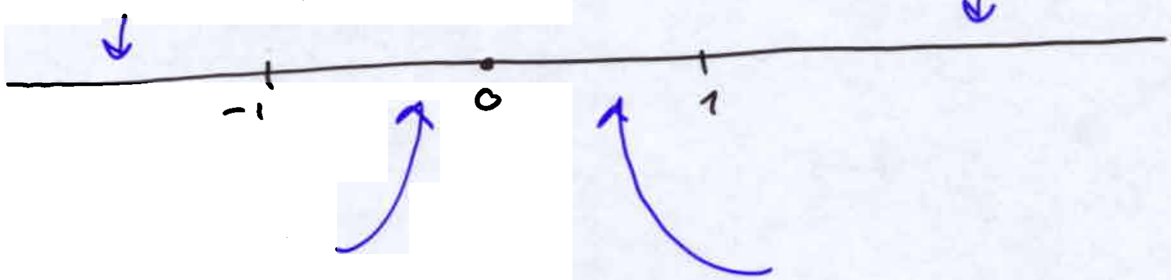
$$\Downarrow$$

$$\boxed{\mathring{\zeta}(s) = \mathring{\zeta}(1-s)}$$

"Non-abelian" ζ - function:

$$\begin{pmatrix} 1 & 2\pi i\,\omega(\lambda)\,\lambda^{s/2-1} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 2\pi i\,\omega(-\lambda)\,(-\lambda)^{\frac{s}{2}-1} & 1 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 0 \\ 2\pi i\,\omega\left(-\frac{1}{\lambda}\right)(-\lambda)^{\frac{s}{2}-3/2} & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2\pi i\,\omega\left(\frac{1}{\lambda}\right)\lambda^{s/2-3/2} \\ 0 & 1 \end{pmatrix}$$