MANIFOLDS WITH SPECIAL HOLONOMY
LECTURE 2:
SOME COMPLEX GEOMETRY

ROBERT L. BRYANT

DUKE UNIVERSITY

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We will begin by considering what is, in some sense, the largest of the special holonomy cases in Berger's list:

Dimension	Group	Invariant forms (generators)
n	$\mathrm{SO}(n)$	$1 \in \Lambda^0,\ *1 \in \Lambda^n$
$n = 2m$	$\mathrm{U}(m)$	$1 \in \Lambda^0,\ \omega \in \Lambda^2$
$n = 2m$	$\mathrm{SU}(m)$	$1 \in \Lambda^0,\ \omega \in \Lambda^2,\ \phi, \psi \in \Lambda^m$
$n = 4m$	$\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$	$1 \in \Lambda^0,\ \Phi \in \Lambda^4$
$n = 4m$	$\mathrm{Sp}(m)$	$1 \in \Lambda^0,\ \omega_1, \omega_2, \omega_3 \in \Lambda^2$
$n = 7$	G_2	$1 \in \Lambda^0,\ \phi \in \Lambda^3,\ *\phi \in \Lambda^4$
$n = 8$	$\mathrm{Spin}(7)$	$1 \in \Lambda^0,\ \Phi \in \Lambda^4$

1. Unitary Holonomy: Endow $\mathbb{R}^{2n}$ with its standard inner product and the orthogonal complex structure

$$J_n = \begin{pmatrix} 0_n & -\mathrm{I}_n \\ \mathrm{I}_n & 0_n \end{pmatrix}.$$

Define the unitary group $\mathrm{U}(n) \subset \mathrm{SO}(2n)$ and embed into $\mathrm{GL}(n, \mathbb{C})$ via

$$\mathrm{U}(n) = \left\{ A \in \mathrm{SO}(2n) \,\middle|\, AJ_n = J_n A \right\} \ni \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \longmapsto a + \mathrm{i}b.$$

If (M^{2n}, g) has holonomy conjugate to a subgroup of $\mathrm{U}(n)$, then M possesses a g-parallel, orthogonal almost-complex structure $J : TM \to TM$. Corresponding to this, there is also a g-parallel 2-form ω related to the almost complex structure by

$$\omega(v, w) = g(Jv, w)$$

In fact, via this equation, any two of (g, J, ω) serve to determine the third.

The fact that J and ω are parallel w.r.t. g implies

(1) ω is closed: $\mathrm{d}\omega = 0$

(2) J is integrable: Each point of M has a coordinate neighborhood (U, z) with $z : U \to \mathbb{C}^n$ a coordinate system satisfying $\mathrm{d}z(Jv) = \mathrm{i}\,\mathrm{d}z(v)$ for all $v \in TU$.

Proof sketch: We already saw that ω is closed.

The integrability of J follows from the Newlander-Nirenberg Theorem. The point is that, in geodesic coordinates $x = (x^i)$ centered on $m \in M$, we have

$$J = J^k_l(x)\,\frac{\partial}{\partial x^k} \otimes \mathrm{d}x^l \quad \text{and} \quad (\nabla J)_m = \frac{\partial J^k_l}{\partial x^j}(0)\,\mathrm{d}x^j \otimes \frac{\partial}{\partial x^k} \otimes \mathrm{d}x^l.$$

The Nijnhuis tensor of J (which, by NNT, obstructs integrability) is linear in the first partials of J in any coordinate system. Thus, $\nabla J = 0$ implies that J is integrable.

Still, there remains the question: How many such metrics can there be?

Say that a pair (J, ω) defined on M are *compatible* if

(1) $\omega(v, Jv) > 0 \quad \forall\, 0 \neq v \in TM$, and
(2) $\omega(v, Jw) = \omega(w, Jv) \quad \forall\, x \in M,\ v, w \in T_xM$.

In this case, we say that the metric g defined by

$$g(v, w) = \omega(v, Jw)$$

is the *associated* metric.

Proposition: If (J, ω) are a compatible pair on M^{2n} and J is integrable and ω is closed, then J and ω are g-parallel, where g is the associated metric. In particular, the holonomy of g is conjugate to a subgroup of $\mathrm{U}(n)$.

Proof sketch: Since J is integrable, M has an atlas of J-holomorphic charts: (U, z) with $z : U \to \mathbb{C}^n$ where $\mathrm{d}z(Jv) = \mathrm{i}\,\mathrm{d}z(v)$. Then (2) above and the definition of g imply that

$$\omega_U = \tfrac{1}{2}\mathrm{i}\, h_{j\bar{k}}\, \mathrm{d}z^j \wedge \mathrm{d}\overline{z^k} \qquad \text{and} \qquad g_U = h_{j\bar{k}}\, \mathrm{d}z^j \circ \mathrm{d}\overline{z^k}$$

for functions $h_{j\bar{k}} = \overline{h_{k\bar{\jmath}}}$ on U with $h = (h_{j\bar{k}}) > 0$ (by (1) above).

Finally, the closure of ω implies that (at least locally), there exists a function f on U so that

$$h_{j\bar{k}} = \frac{\partial^2 f}{\partial z^j \partial \overline{\overline{z^k}}}.$$

Conversely, starting with *any* smooth 'potential' function f on a domain $D \subset \mathbb{C}^n$ such that the quadratic form

$$g = \frac{\partial^2 f}{\partial z^j \partial \overline{\overline{z^k}}} \, \mathrm{d}z^j \circ \mathrm{d}\overline{z^k}$$

is positive definite on D, one computes that the standard complex structure J on $\mathbb{C}^n$ and the 2-form

$$\omega = \frac{\mathrm{i}}{2} \frac{\partial^2 f}{\partial z^j \partial \overline{\overline{z^k}}} \, \mathrm{d}z^j \circ \mathrm{d}\overline{z^k}.$$

are parallel with respect to the Levi-Civita connection of g. (Hint: to simplify the calculations, add the real part of a holomorphic function of z to f and choose holomorphic coords $w = (w^j)$ so that $f = |w|^2 +$ (terms vanishing to order ≥ 4).)

Consequences:

(1) Metrics in dimension $2n$ with unitary holonomy exist and depend on one 'arbitrary' function of $2n$ variables, up to diffeomorphism.
(2) The 'generic' such metric has holonomy equal to $\mathrm{U}(n)$.

The data (M, g, J, ω) as above (with the integrability conditions assumed) is said to define a *Kähler structure*. Such manifolds have been extensively studied, in large part because these are the natural metrics one would like to use in studying complex manifolds (M, J).

A very fundamental example is the Riemannian symmetric space

$$\mathbb{CP}^n = \mathrm{SU}(n{+}1)/\mathsf{S}\big(\mathrm{U}(1)\times\mathrm{U}(n)\big),$$

i.e., complex projective space. Up to constant multiples, there is only one $\mathrm{SU}(n{+}1)$-invariant metric g on this space and it has holonomy isomorphic to $\mathrm{U}(n)$, so there is a corresponding invariant complex structure J and 2-form ω.

Important Properties of Kähler manifolds: (M, g, J, ω)

(1) Every complex submanifold $N \subset M$ inherits a Kähler structure just by pullback. (In particular, the smooth points of an algebraic variety $V \subset \mathbb{CP}^n$ inherit a Kähler structure.)

(2) (Wirtinger) The forms $\phi_p = \frac{1}{p!}\omega^p$ are *calibrations* on M with respect to the metric g. In particular, a compact, complex p-dimensional subvariety $N \subset M$ minimizes volume in its homology class.

(3) (Hodge decomp.) There is a ∇-parallel splitting

$$\Lambda^k(T^*M) \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}(T^*M) \qquad \text{(as } J\text{-eigenspaces)}$$

and a decomp. of $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ as $\mathrm{d} = \partial + \bar{\partial}$ with

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$$
$$\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M).$$

n	$\mathfrak{h} \subseteq \mathfrak{so}(n)$	$K(\mathfrak{h})$ as an $\mathfrak{h}$-module
n	$\mathfrak{so}(n)$	$\mathbb{R} \oplus \mathsf{S}^2_0(\mathbb{R}^n) \oplus W_n(\mathbb{R}^n)$
$n = 2m > 2$	$\mathfrak{u}(m)$	$\mathbb{R} \oplus \mathsf{S}^{1,1}_0(\mathbb{C}^m)^{\mathbb{R}} \oplus \mathsf{S}^{2,2}_0(\mathbb{C}^m)^{\mathbb{R}}$
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$n = 4m > 4$	$\mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$	$\mathbb{R} \oplus \mathsf{S}^4(\mathbb{C}^{2m})^{\mathbb{R}}$
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$n = 7$	$\mathfrak{g}_2$	$\mathsf{V}^{0,2} \simeq \mathbb{R}^{77}$
$n = 8$	$\mathfrak{spin}(7)$	$\mathsf{V}^{0,2,0} \simeq \mathbb{R}^{168}$

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2. Special Unitary Holonomy. Let

$$\mathrm{SU}(n) = \{\ A \in \mathrm{U}(n) \mid \det_{\mathbb{C}} A = 1\}.$$

Note that $\mathrm{SU}(n)$ can be characterized as the subgroup of $\mathrm{GL}(2n, \mathbb{R})$ that preserves the exterior forms

$$\begin{aligned} \omega_0 &= \tfrac{1}{2}\,\mathrm{i}\left(\mathrm{d}z^1 \wedge \mathrm{d}\overline{z^1} + \cdots + \mathrm{d}z^n \wedge \mathrm{d}\overline{z^n}\right) \\ \Upsilon_0 &= \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \cdots \wedge \mathrm{d}z^n. \end{aligned}$$

In fact, the algebra of $\mathrm{SU}(n)$-invariant exterior forms on $\mathbb{C}^n$ is generated by ω_0, Υ_0, and $\overline{\Upsilon_0}$.

Conversely, let V be a (real) v. s. of dim. $2n$ and let ω and Υ be exterior forms of degrees 2 and n, respectively, on V such that

1. Υ is decomposable and satisfies $\Upsilon \wedge \overline{\Upsilon} \neq 0$

Then there is a unique complex structure $J : V \to V$ so that Υ spans $\Lambda^{n,0}(V)$. Suppose now further that ω is real-valued and

2. $\omega(v, Jv) > 0$ for $v \neq 0$ and $\omega \in \Lambda^{1,1}(V)$.

Then there is a constant $\lambda > 0$ such that $(\omega, \Upsilon) \simeq (\omega_0, \lambda\Upsilon_0)$.

Now suppose that (M^{2n}, g) has holonomy conjugate to a subgroup of $\mathrm{SU}(n) \subset \mathrm{SO}(2n)$. Then, in addition to having a parallel complex structure J and 2-form ω, the manifold will support a g-parallel $\mathbb{C}$-valued n-form $\Upsilon \in \Omega^{n,0}(M)$. Multiplying Υ by a positive constant, we can suppose

$$3.\ \ \Upsilon \wedge \overline{\Upsilon} = \frac{2^n}{\mathrm{i}^{n^2}\, n!}\, \omega^n.$$

Of course, since Υ is g-parallel, it must be closed (and co-closed).

A pair (ω, Υ) of closed forms on a $2n$-manifold M satisfying (pointwise) the conditions (1–3) listed above constitute a *Calabi-Yau structure* on M.

Proposition: For any Calabi-Yau structure (ω, Υ) on M^{2n}, the associated almost complex structure J is integrable and the forms ω and Υ are parallel with respect to the associated metric g.

Proof sketch: By hypothesis, locally $\Upsilon = \zeta^1 \wedge \zeta^2 \wedge \cdots \wedge \zeta^n$ where the ζ^k and the $\overline{\zeta^k}$ are linearly independent.

Now, J is defined by $\zeta^k(Jv) = \mathrm{i}\,\zeta^k(v)$ for $v \in TM$ and the hypothesis $\mathrm{d}\Upsilon = 0$ implies that J is integrable and that Υ is actually holomorphic w.r.t. J. In fact, it is possible to choose local J-holomorphic coordinates (U, z) so that

$$\Upsilon_U = \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \cdots \wedge \mathrm{d}z^n = \mathrm{d}z.$$

We have already seen that the integrability of J and the closure of ω imply that, locally, there is a function f on U such that

$$\omega_U = \frac{\mathrm{i}}{2}\,\frac{\partial^2 f}{\partial z^j\,\partial\overline{z^k}}\,\mathrm{d}z^j \wedge \mathrm{d}\overline{z^k} \qquad \text{and} \qquad g_U = \frac{\partial^2 f}{\partial z^j\,\partial\overline{z^k}}\,\mathrm{d}z^j \circ \mathrm{d}\overline{z^k}$$

The hypotheses on ω and Υ then imply

$$\left(\frac{\partial^2 f}{\partial z^j\,\partial\overline{z^k}}\right) > 0 \qquad \text{and} \qquad \det\left(\frac{\partial^2 f}{\partial z^j\,\partial\overline{z^k}}\right) = 1.$$

Now one can compute that, if f is any smooth function on a domain $D \subset \mathbb{C}^n$ that satisfies these two conditions, then g as as metric on D not only leaves ω and J parallel, but $\Upsilon = \mathrm{d}z$ as well. Thus, the holonomy of such a metric lies in $\mathrm{SU}(n)$.

Local Properties: Let (ω, Υ) be a Calabi-Yau structure on M^{2n}.

(1) The general Calabi-Yau structure depends on 2 functions of $2n{-}1$ variables (modulo diffeomorphism) and is real-analytic in J-holomorphic coordinates.

(2) The associated metric g of the 'generic' Calabi-Yau structure on M^{2n} has holonomy equal to $\mathrm{SU}(n)$.

(3) The associated metric g is Ricci-flat. Conversely, if M is simply connected and (M, g, J, ω) is a Kähler structure for which g is Ricci-flat, then g admits a parallel holomorphic volume form Υ such that (ω, Υ) is Calabi-Yau.

(4) (Harvey-Lawson) The real-valued n-form $\phi = \mathrm{Re}(\Upsilon)$ is a *calibration* on (M, g) (called the *special Lagrangian calibration*). It calibrates a large family of Lagrangian submanifolds of M. In fact, any real-analytic $(n{-}1)$-dimensional submanifold $P \subset M$ that satisfies $P^*\omega = 0$ (i.e., is sub-Lagrangian) lies in a unique (local) analytically irreducible Lagrangian submanifold $N \subset M$ that is calibrated by ϕ.

Global Existence.

Calabi's Complete Example. Idea: Look for a rotationally invariant metric on $\mathbb{C}^n$ with $\Upsilon = \mathrm{d}z$, i.e., with

$$\omega = \frac{\mathrm{i}}{2}\,\partial\bar\partial\big(f(|z|^2)\big).$$

Now, $\omega > 0$ implies $f'(\rho) > 0$ and $\rho f''(\rho) + f'(\rho) > 0$ ($\rho = |z|^2$). The volume relation between ω^n and $\Upsilon \wedge \overline{\Upsilon}$ becomes the ODE

$$f'(\rho)^{n-1}\big(\rho f''(\rho) + f'(\rho)\big) = 1.$$

This has a first integral

$$\big(\rho\, f'(\rho)\big)^n = \rho^n + c.$$

If $c = 0$, this is the flat metric. If $c > 0$, this says that

$$\omega = \frac{\mathrm{i}}{2}\,\partial\bar\partial\big(f(|z|^2)\big) = \frac{\mathrm{i}}{2}\,\partial\left(\frac{(|z|^{2n} + c)^{1/n}}{|z|^2}\, z \cdot \mathrm{d}\bar z\right),$$

but this metric is singular at $z = 0$ if $n > 1$!

However, this singularity can be resolved: Blow up $\mathbb{C}^n$ at the origin

$$\widehat{\mathbb{C}^n} \longrightarrow \mathbb{C}^n$$

where

$$\widehat{\mathbb{C}^n} = \big\{([w], z) \,\big|\, w \neq 0, z = \lambda w \ \big\},$$

and then divide by the $\mathbb{Z}_n$ action $\big([w], z\big) \cdot \lambda = \big([w], \lambda z\big)$ $(\lambda^n = 1)$. The resulting space

$$X_n = \widehat{\mathbb{C}^n}/\mathbb{Z}_n$$

turns out to be a smooth n-manifold, namely $\Lambda^{n,0}(\mathbb{CP}^{n-1})$, the canonical bundle of $\mathbb{CP}^{n-1}$.

Now, ω lifts up to $\widehat{\mathbb{C}^n}$ to become a smooth 2-form $\hat{\omega}$ that's degenerate on the blow-up divisor. However, it then pushes down to X_n to be a smooth and positive $(1,1)$-form $\tilde{\omega}$.

Finally Υ goes along for the ride to induce a holomorphic volume form $\tilde{\Upsilon}$ on X_n.

The resulting $(\tilde{\omega}, \tilde{\Upsilon})$ is a Calabi-Yau structure whose underlying metric $\tilde{g}$ on X_n is complete. This is Calabi's example.

Compact examples. Calabi conjectured and S.T. Yau proved the following existence result that supplies lots of examples of metrics with holonomy in $\mathrm{SU}(n)$ (in theory).

Theorem: Let M be a compact complex n-manifold that admits a holomorphic volume form Υ and a Kähler form ω_0. Then there exists a function f on M and a real constant $\lambda > 0$ so that

$$\big(\,\omega_0 + \mathrm{i}\partial\bar{\partial} f,\ \lambda\Upsilon\,\big)$$

is a Calabi-Yau structure on M.

Example: If $X_n \subset \mathbb{CP}^{n+1}$ is a smooth hypersurface of degree $n+2$, it has a nonvanishing holomorphic n-form. It certainly has a Kähler form, just pull back the Kähler form on $\mathbb{CP}^{n+1}$. Thus, by the Calabi-Yau Theorem, such an X_n carries a Calabi-Yau structure. It can be shown that this can never be a product and, in fact, the holonomy is always equal to $\mathrm{SU}(n)$.

When $n = 2$, the quartic surface $X_2 \subset \mathbb{CP}^3$ is one of the famous K3 surfaces. Its Calabi-Yau metrics are still not explicitly known.