Lawson: Lecture I

In differential geometry, the geometry is defined using a manifold and a tensor. The Greeks thought of geometry in terms of objects - lines, planes,... Modern geometry can be done this way too.

A riemannian manifold has natural submanifolds - the geodesics.

In algebraic geometry one has the subvarieties and the scheme approach uses the subvarieties as the points in the total space.

Question: What are the natural subvarieties of the exceptional geometries?

(Aside: we will restrict to 1^{st} order conditions.)

Let X be a smooth manifold, we have the Grassman Bundle of oriented tangent p-planes

$$G_p(TX) \longrightarrow X.$$

Let $\mathcal{G} \subset G_p(TX)$ be any subset.

<u>Def:</u> An oriented C^1-submanifold $M \subset X$ is a $\mathcal{G}$-manifold if $\vec{T}_x M \in \mathcal{G} \quad \forall x \in M$ ($\vec{T}_x M$ means the oriented tangent plan)

Ex.'s: If X is complex we have the $\mathbb{C}$-submanifolds.
If X is symplectic we have Lagrangian submanifolds.

Note: Geodesics are 2^{nd} order, as are minimal surfaces
We'll consider 1^{st} order conditions which $\Rightarrow$ minimality.

V $\mathbb{R}$- vector space with inner product $\langle\, ,\rangle$.

$$G_p(V) \equiv \{\text{oriented } p\text{-dim'l linear subspaces of } V\}$$

$\Lambda^p V$ has a natural innerproduct. On simple vectors

$$\langle v_1\wedge\dots\wedge v_p, w_1\wedge\dots\wedge w_p\rangle \equiv \det(\langle v_i, w_j\rangle)$$

$\uparrow$ (this is skew symmetric and multilinear)

and then extend linearly.

Let $|v_1\wedge\dots\wedge v_p|^2 = \langle v_1\wedge\dots\wedge v_p, v_1\wedge\dots\wedge v_p\rangle$

Basic Fact: $\exists$ smooth embedding

$$G_p(V) \hookrightarrow \text{unit sphere in } \Lambda^p V$$

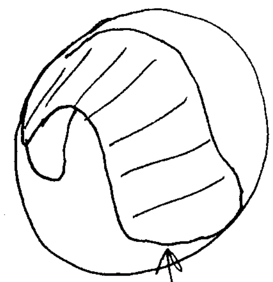
$$\begin{array}{ccc} \cup\!\!\!\shortmid & & \\ Q & \longmapsto & \dfrac{v_1\wedge\dots\wedge v_p}{|v_1\wedge\dots\wedge v_p|} \end{array}$$

for $v_1,\dots,v_p$ any oriented basis of Q.

If $w_1,\dots,w_p$ is another basis of Q $\quad w_j = \sum_{k=1}^{p} a_{jk} v_k \qquad a_{jk} = (A)_{jk}$

$$w_1\wedge\dots\wedge w_p = \underbrace{\det(A)}_{>0}\, v_1\wedge\dots\wedge v_p$$

Mass Norm on $\Lambda^p V$: unit ball = convex hull of $G_p(V)$.
(not smooth)



extrem pts. are simple vectors

$$\varphi \in (\Lambda^p V)^* = \Lambda^p V^*$$

Def: The comass norm of φ is given as:

$$\|\varphi\| \equiv \sup\{\varphi(\xi) : \xi \in G_p(V)\}.$$

Note: $\|\varphi\| \le |\varphi|$ since $|\varphi| = \sup\{\varphi(\eta) : \eta \in \Lambda^p V, |\eta| = 1\}$

comass Euclidean

Def Suppose $\|\varphi\| = 1$, $\mathcal{G}(\varphi) = \{\xi \in G_p(V) : \varphi(\xi) = 1\}$

"φ - planes".

For generic φ $\mathcal{G}(\varphi)$ is a point. We want φ s.t. $\mathcal{G}(\varphi)$ is large.

Back to manifolds: Let X be riemannian.

$$\begin{tikzcd} G_p(TX) \arrow[dr] & \subset & \Lambda^p TX \arrow[dl] \\ & X & \end{tikzcd}$$

$G_p(TX) \hookrightarrow$ unit sphere bundle of $\Lambda^p TX$.

Using the metric we have the mass norm and comass norms.

Given $\varphi \in \Gamma(\Lambda^p T^* X)$ (at least C^1)

Def: φ is called a calibration if (1) $\|\varphi_x\| \le 1$ $\forall x \in X$

(2) $d\varphi = 0$

$\mathcal{G}(\varphi) \subset G_p(X)$ (in general this is not a fiber bundle)

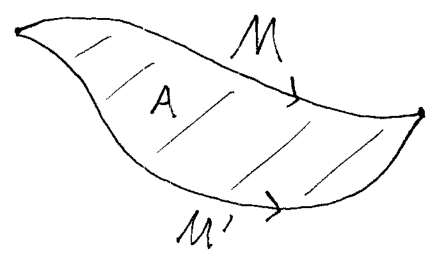
<u>Def:</u> An oriented C^1-submanifold $M \subset X$ is a <u>φ-manifold</u> if $\vec{T}_x M \in \mathcal{G}(\varphi)$ $\forall x \in M$. (or a <u>calibrated submanifold</u>, calibrated by φ.)

Or M is a φ-manifold $\Leftrightarrow$ $\varphi|_M = d\mathrm{vol}_M$ since $\varphi|_M = \varphi(\vec{T}_x M)\, d\mathrm{vol}_M$.

<u>Theorem:</u> If φ is a calibration, then any φ-submanifold is <u>homologically</u> <u>volume</u> <u>minimizing</u>, i.e. if

M is cpt. φ-manifold with boundary, ∂M, and M' is any other cpt. oriented C^1-submanifold of X

s.t. $\partial M' = \partial M$ and $M - M' = \partial A$ for some real $(p+1)$-chain A



(A need not be a manifold)

then $\mathrm{vol}(M) \leq \mathrm{vol}(M')$.

Moreover, if $\mathrm{vol}(M) = \mathrm{vol}(M')$ then M' is also a φ-submanifold.

<u>Proof:</u> $\int_M \varphi - \int_{M'} \varphi = \int_{M-M'} \varphi = \int_{\partial A} \varphi = \int_A d\varphi = 0$

$\varphi|_M = \varphi(\vec{T}_x M)\, d\mathrm{vol}_M = d\mathrm{vol}_M$

$\varphi|_{M'} = \varphi(\vec{T}_x M')\, d\mathrm{vol}_{M'} \leq d\mathrm{vol}_{M'}$

so $\mathrm{vol}(M) = \int_M \varphi = \int_{M'} \varphi = \int_{M'} \varphi(\vec{T}_x M')\, d\mathrm{vol}_{M'} \leq \mathrm{vol}(M')$

$$\boxed{\text{and "=" holds} \Leftrightarrow \varphi(\vec{T}_x M') = 1 \ \forall x \in M'}$$

Consequences: If X is a manifold of Class C^k $2 \le k \le \omega$

then ① M is C^k (follows by regularity theory of Morrey)

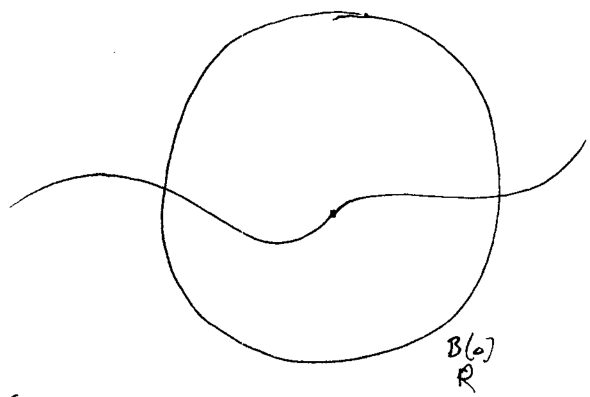
② mean curvature $\equiv 0$

③ $\exists$ Certain Monotonicity Properties:

If $X = \mathbb{R}^n$ $\partial M \cap B_R(0) = \emptyset$

then

$$\frac{\mathrm{vol}(M \cap B_r(0))}{\alpha_p r^p}$$ is



a monotone increasing function of r for $0 \le r \le R$.

$\left(\alpha_p \equiv \mathrm{vol}(B_1^p) \right)$

$$\lim_{r \to 0} \frac{\mathrm{vol}(M \cap B_r(0))}{\alpha^p r^p} = 1 \quad (\text{if } M \text{ contains } 0 \in \mathbb{R}^n)$$

First Case Kähler Geometry.

$W = \mathbb{C}$ - vector space with hermitian i-product

(i) $J : W \to W \quad J^2 = -id$

(ii) $\langle \cdot , \cdot \rangle$ $\mathbb{R}$, pos. inner product

$$\langle Jv, Jw \rangle = \langle v, w \rangle \quad \forall v, w \in W.$$

$$\omega(v, w) \equiv \langle Jv, w \rangle$$

Prop (Wirtinger $\leq$) Let $Q \subset W$ be any oriented $\mathbb{R}$-linear subspace of dimension $2p$

$$(Q \in G_{2p}(W))$$

$\frac{\omega^p}{p!}(Q) \leq 1$ and $= 1 \Leftrightarrow Q$ is a canonically oriented $\mathbb{C}$-subspace of $\mathbb{C}$-dimension p.

i.e. $\frac{\omega^p}{p!}$ has comass 1 and

$$\mathcal{G}\left(\frac{\omega^p}{p!}\right) = G_p^{\mathbb{C}}(W) \subset G_{2p}^{\mathbb{R}}(W)$$

Pf $\omega|_Q$ is skew symm. so $\exists$ an oriented orthonormal basis $\{e_1, f_1, \dots, e_p, f_p\}$ of Q

s.t. $\omega \cong \begin{pmatrix} \begin{matrix} 0 & \lambda_1 \\ \lambda_1 & 0 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} 0 & -\lambda_p \\ \lambda_p & 0 \end{matrix} \end{pmatrix}$, $\omega|_Q = \sum_{j=1}^{p} \lambda_j e_j^* \wedge f_j^*$

$$(\omega|_Q)^p = p!(\lambda_1 \cdots \lambda_p)\, e_1^* \wedge f_1^* \wedge \cdots \wedge e_p^* \wedge f_p^*$$
$$= p!(\lambda_1 \cdots \lambda_p) \cdot d\mathrm{vol}_Q$$

We can assume that $\lambda_2 \geq 0, \dots, \lambda_p \geq 0$

$$\lambda_j = \omega(e_j, f_j) = \langle Je_j, f_j \rangle \leq |Je_j||f_j| = 1$$

Get equality $\iff Je_j = f_j \quad \forall j$.

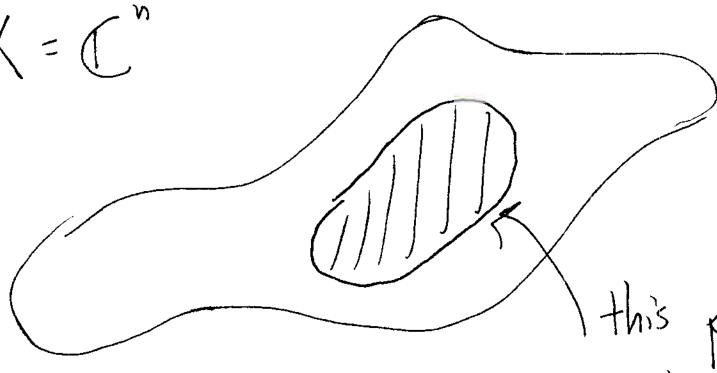
Suppose $(X, \langle\, ,\rangle, J, \omega)$ = Kähler manifold.

Then $\frac{\omega^p}{p!}$ is a calibration.

$$\mathcal{G}\left(\frac{\omega^p}{p!}\right) = G_p^{\mathbb{C}}(TX) \subset G_{2p}^{\mathbb{R}}(TX).$$

so $\frac{\omega^p}{p!}$ - submanifolds are the $\mathbb{C}$-submanifolds and any $\mathbb{C}$-sub. man. of a Kähler manifold is homol'ly volume minimizing

Ex: $X = \mathbb{C}^n$



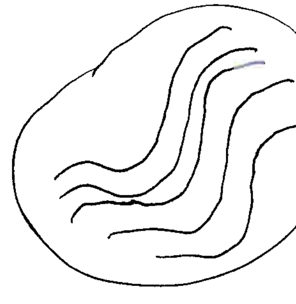
this piece is ~~the~~ volume minimizing and since 2-complex submanifolds which share a boundary are identically, it is unique.

Ex: $X = \mathbb{P}^n_{\mathbb{C}}$ $\quad H_{2p}(X, \mathbb{Z}) = \mathbb{Z}[\mathbb{P}^p_{\mathbb{C}}]$

so if M, is a compact complex submanifold of $\mathbb{P}^n_{\mathbb{C}}$ $\partial M = \emptyset$, then $\mathrm{Vol}(M) \leq \mathrm{vol}(M') \;\; \forall M'$ in some homology class.

So all the $\mathbb{C}$-manifolds in a given homology have the same volume, even though they may look very different.

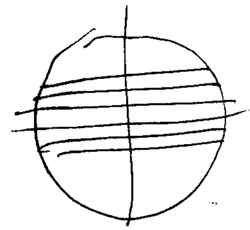
Foliations



X riem. manifold, $\dim_{\mathbb{R}} X = n+1$

$\mathcal{F}$ = smooth codim-1 foliation.

choosing new coordinate



Assume $\mathcal{F}$ oriented.

φ = unit volume form of $\mathcal{F}$.

Locally $e_1, \dots, e_n$ an O.N. frame field for $T\mathcal{F}$

$$\varphi = e_1^* \wedge \dots \wedge e_n^* \cong e_1 \wedge \dots \wedge e_n$$

$$= * \nu \qquad \nu = \text{unit normal}$$

$\| \varphi \| = |\varphi| \equiv 1$

Lemma $\quad d\varphi = -H \, d\mathrm{vol}_X = -H \, e_1 \wedge \dots \wedge e_n \wedge \nu$, $\quad H$ = mean curvature of $\mathcal{F}$

so φ is a calibration $\Longleftrightarrow$ the leaves of $\mathcal{F}$ are minimal.

Proof:
$$d\varphi = \sum_{j=1}^{n} e_j \wedge \nabla_{e_j} \varphi + \nu \wedge \nabla_\nu \varphi$$

$$= \sum_{j,k} e_j \wedge (e_1 \wedge \dots \wedge (\nabla_{e_j} e_k) \wedge \dots \wedge e_n)$$

$$+ \nu \wedge \sum_{j=1}^{n} e_1 \wedge \dots \wedge \nabla_\nu e_j \wedge \dots \wedge e_k$$

$$\text{so } d\varphi = \sum_j e_j \wedge \left(e_1 \wedge \cdots \wedge [\nabla_{e_j} e_j]^N \wedge \cdots \wedge e_n \right)$$

$$+ \nu \wedge \sum_{j=1}^{n} e_1 \wedge \cdots \wedge \cancel{\langle \nabla_\nu e_j, e_j \rangle}^{\,0} e_j \wedge \cdots \wedge e_n$$

$$= \underbrace{- \sum_{j=1}^{n} \langle \nabla_{e_j} e_j, \nu \rangle}_{H} \, \nu \wedge e_1 \wedge \cdots \wedge e_n \qquad \left(\langle \nabla_\nu e_j, e_j \rangle = \tfrac{1}{2} \nu \|e_j\|^2 \right)$$

$H = \hookleftarrow$

If $H \equiv 0$, every compact leaf is hom. minimizing

so $\neq 0$ in $H_{n-1}(X, \mathbb{R})$.

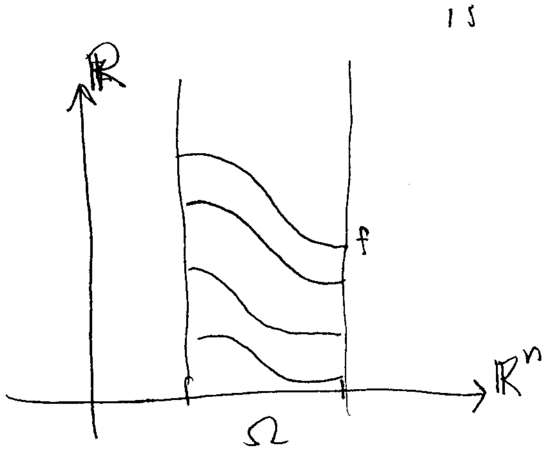
If $\mathcal{F}$ has a compact leaf $\sim_{\mathbb{R}} 0$,

$\nexists$ metric s.t. $H \equiv 0$..

$\Omega^{\text{open}} \subset \mathbb{R}^n$. $f : \Omega \to \mathbb{R}$

Assume $gr(f) = \{ (x, f(x)) \in \mathbb{R}^n \times \mathbb{R} : x \in \Omega \}$

is a minimal hypersurface $H \equiv 0$



so $\Omega \times \mathbb{R}$ is foliated

by $gr(f+c)$ $c \in \mathbb{R}$.

Interesting Case: $\Omega = \mathbb{R}^n$

$gr(f)$ is h.v. minimizing $\therefore$ absolutely v-minimizing.

No need to rule out singularities. For the form $\frac{\omega^p}{p!}$ one can allow complex analytic subvarieties (which have singularities).

Rectifiable Set: a set is rectifiable in dimension p if it can be written $E_0 \amalg \left(\coprod_{k=1}^{\infty} E_k \right)$

$\mathcal{H}^p(E_0) = 0$ (hausdorff measure)

$E_k \subset M_k = C^1$ - sub. of dim p

Nice Properties: 1) Rect. sets have compactness properties

2) they have tangent planes at every point

can study φ - rectifiable sets

φ - rectifiable currents