Lecture three Chern classes and Donaldson's functionals

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Let $E \to X$ be a complex vector bundle over a compact Hermitian manifold X. Then for any connection D. Let $\Omega = D^2$ be the curvature. Then Ω is a matrix-valued 2-form. Define

$$c(E) = \det\left(\frac{\sqrt{-1}}{2\pi}(I+\Omega)\right)$$

to be the total Chern class.

Definition of the Grothendieck ring $G(X)$: Let ξ be the set of all vector bundles over X. Then under the binary operatio $\oplus$ and $\otimes$, it is a ... (with $\oplus$ it is a monoid). Let $G(X)$ be the Abelian group generated by ξ. Then under $\otimes$ $G(X)$ be a commutative ring. The ring is call the Grothendieck ring.

An element of $G(X)$ can be written as $E_1 - E_2$ formally, where $E_1, E_2 \in \xi$. $E_1 - E_2 = E_3 - E_4$, if there is an E_5 such that $E_1 + E_4 + E_5 = E_2 + E_3 + E_5$. By the Whitney formula for Chern classes, we have

$$c(E_1)\, c(E_4)\, c(E_5) = c(E_2)\, c(E_3)\, c(E_5)$$

Thus

$$c(E_1)/c(E_2) = c(E_3)/c(E_4)$$

~~Because of this, we have the following~~

We need to define the following characteristic classes

1. Chern character

We consider the invariant polynomial (in this case, it is an invariant analytic function)

$$Ch(x_1, \cdots, x_r) = \sum e^{x_i}$$

Define

$$Ch(E) = Ch(\Omega)$$

to be the Chern character.

2. Ch_n

Define

$$Ch_n(x_1, \cdots, x_r) = (x_1^n + \cdots + x_r^n)/n!$$

Then

$$Ch_n(E) = Ch_n(\Omega)$$

3. Todd Class

Define

$$Td(x_1, \cdots, x_r) = \prod_i \frac{x_i}{1-e^{-x_i}}$$

Then the ~~Td~~ Todd class is defined as

$$Td(E) = Td(\Omega).$$

We have the following results.

① $c(E \oplus F) = c(E) \cdot c(F)$

② $Ch(E \oplus F) = Ch(E) + Ch(F)$; $ch(E \otimes F) = ch(E) \cdot ch(F)$

③. $Td(E \oplus F) = Td(E) \cdot Td(F)$.

Before going further, we would like to give an example as an exercise to the Chern classes and the Hodge decomposition Theorem.

Example Let X be a smooth 4-th order ~~poly~~ hyper-surface of $\mathbb{CP}^3$. X is a K3 surface. Compute the dimension $\dim H^1(X, TX)$ of universal deformation space.

Solution 1. Using Serre duality we have

$$H^1(X, TX) = H^{1,1}(X)$$

By Leschefsctz ~~theorem~~ hyperplane section Theorem

$$\dim H^1(X) = 0.$$

The Euler characteristic number

$$\chi = 1 + \dim H^2(X) + 1$$

By the Hodge de composition theorem

$$\dim H^2(X) = 2 + \dim H^{1,1}(X)$$

Thus

$$\chi = 4 + \dim H^1(X, TX)$$

We are going to prove that $\chi = 24$. By Gauss-Bonnett

$$\chi = \int_X c_2(X)$$

On the other hand, we have

$$0 \to TX \to T\mathbb{CP}^3|_X \to N \to 0$$

Thus

Euler sequence.

$$\boxed{0 \to \mathbb{C} \to H^{n+1} \to T'(\mathbb{P}) \to 0}$$

$$c(T\mathbb{CP}^n|_X) = c(N)\,c(TX)$$

N is a line bundle. So we have

$$C(N) = 1 + C_1(N)$$

X is a C-Y. Thus

$$c(X) = 1 + C_2(X)$$

$$c(T\mathbb{CP}^n|_X) = (1+\omega)^4 = \cdots + 6\omega^2 + \cdots$$

Comparing both sides, we have

$$C_2(X) = 6\omega^2$$

Thus

$$\chi = \int C_2(X) = 6\int \omega^2 = 24$$

Thus

$$\dim H^1(X, TX) = 20$$

Solution 2. Suppose $f = \sum a_{i_0 \cdots i_3} Z_0^{\alpha_0} \cdots Z_3^{\alpha_3}$ be a 4-th order equation. The space f is of dimension N where N is the number of solutions of the equation

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 4, \quad \alpha_i \geq 0$$

A straightforward computation gives $N = 35$. Thus The space of Hilbert scheme is ~~34~~ of dimension 34. The $\dim \mathrm{Aut}(\mathbb{CP}^3) = 15$. Thus for K3 surfaces which are hypersurfaces of $\mathbb{CP}^3$. The dimension is 19.

The result is not surprising because ω ~~dete~~ determine a deformation that is not with respect to the same polarization.

Now let's turn to the Bott-Chern class. Assume now that E is a Hermitian bundle over X. That is, E is a holomorphic vector bundle with a Hermitian metric h. Let h_0 be a fixed Hermitian metric h_0. Let φ be any invariant polynomial. Then by the $\partial\bar\partial$-Poincaré Lemma, we know that

$$\varphi(R(h)) - \varphi(R(h_0)) = \partial\bar\partial \xi$$

for some $(k-1, k-1)$ form. In what follows, we define the form ξ.

Let (h_0, h_1) be two metrics on the Hermitian vector bundle E. Let h_t be a smooth curve connecting h_0 and h_1. For example, $h_t = (1-t)h_0 + t h_1$. Let φ be an invariant polynomial. Then

Lemma (Donaldson)

$$\int_0^1 \tilde{\varphi}(\dot{h}_t h_t^{-1}, R(h_t), \cdots R(h_t))dt$$

$$\in \operatorname{Im}\partial \oplus \operatorname{Im}\bar\partial$$

In particular, up to the subspace of $\operatorname{Im}\partial \oplus \operatorname{Im}\bar\partial$, the integral is independent of the choice of the path connecting h_0 and h_1.

Definition Let K be the space of Hermitian metrics of E. Define a functional

$$BC(\varphi, \cdot, \cdot) : K \times K \longrightarrow \Omega^{k-1,k-1} / Im(\partial) \oplus Im\,\bar\partial$$

by

$$BC(\varphi, h_0, h_1) = \int_0^1 k\tilde{\varphi}(\dot{h}_t h_t^{-1}, R(h_t), \cdots, R(h_t))\,dt$$

Then $BC(\varphi, h_0, h_1)$ is called the Bott-Chern classes or the second characteristic class.

The following properties of Bott-Chern classes are essential:

1. $BC(\varphi, h, h) = 0$, $BC(\varphi, h_0, h_1) + BC(\varphi, h_1, h_2) = \cancel{\varphi(}\ BC(\varphi, h_0, h_2)$
2. $BC(\varphi_1 + \varphi_2, h_1, h_2) = BC(\varphi, h_1, h_2) + BC(\varphi_2, h_1, h_2)$
3. $\bar\partial\partial BC(\varphi, h_1, h_2) = \varphi(R(h_2)) - \varphi(R(h_1))$

As in the case of Chern classes, we can extend the Bott-Chern classes into virtual bundles. In view of the fact that

$$Ch(E \oplus F) = Ch(E) + Ch(F)$$
$$Ch_n(E \oplus F) = Ch_n(E) + Ch_n(F)$$

We make the following definition:

Suppose $h_0 = h_{0,1} - h_{0,2}$, $h_1 = h_{1,1} - h_{1,2}$ are two virtual metrics. We define

$\cancel{Ch_k\,C}$

$$BC(Ch_k, h_0, h_1) = BC(Ch_k, h_{0,1}, h_{1,1}) - BC(Ch_k, h_{1,1}, h_{1,2})$$

In general, if φ is an invariant polynomial. Then there is a polynomial f such that

$$\varphi = f(ch_1, \cdots, ch_r)$$

We define

$$BC(\varphi, h_0, h_1) = \sum_i \int_0^1 \frac{\partial f}{\partial ch_i}\left(ch_1(R(h_t)), \cdots ch_n(R(h_t))\right)$$

$$\times\, i\left[ch_i(\dot{h}_{t,1} h_{t,1}^{-1}, \cdots R(h_{t,1})) - ch_i(\dot{h}_{t,2} h_{t,2}^{-1}, \cdots R(h_{t,2}))\right]$$

Donaldson's functionals

Let $\varphi_1, \cdots \varphi_J$ be $k_1 \cdots k_J$ homogeneous invariant polynomial. Let h_0, h_1 be two metrics on the vector bundle E. Then

$$D(\varphi_1, \cdots \varphi_J, h_0, h_1) = \sum_{\alpha=1}^{J} \int_X BC(\varphi_\alpha, h_0, h_1) \wedge \omega^{n-k_\alpha+1}$$

where ω is a fixed Kähler form on X.

By the properties ~~to~~ of the Bott-Chern classes, we know that Donaldson functional is well-defined and its critial point is independent of the choice of the initial metric h_0.

Remark: Donaldson's functional can be naturally extended to virtual bundles.

Let $Q_i = Ch_i$. Then the Euler-Lagrange equation of h is

$$\sum_{\alpha=1}^{I} \int_X \frac{\partial f}{\partial Q_\alpha}(Q_1, \cdots, Q_r)\,[\,Q_i'(u_1 h_1^{-1}, R(h_1))$$
$$- Q_i'(u_2 h_2^{-1}, R(h_2))\,] \wedge \omega^{n-k_\alpha+1} = 0$$

In what follows we shall see the restricted Donaldson's functional and its relations to K-E geometry.

Example Let X be a compact Kähler manifold with positive first Chern class. Such a manifold is called a Fano manifod. It is polarized by the anti-canonical bundle $K_X^{-1} \to X$. Let

$$E = (K_X^{-1} - K_X)^{n+1}$$

Let K be the set of all Kähler metrics in the cohomological class $C_1(X)$. Then for any $\omega \in K$, there is a Hermitian metric h such that $-\sqrt{-1}\,\partial\bar\partial \log h = \omega$. h is unique up to a constant. Since h, as a metric on the anti-canonical bundle, is just a volume form, we can fix h_ω by assuming

$$\int_X h_\omega = \int_X \omega^n$$

We define the Donaldson's functional on the set of Hermitian metrics h_ω such that

① $\int_X h_\omega = \int_X \omega^n$
② $Ric(h_\omega) > 0$

by

$$G(h) = D(Ch_{n+1}, h_0, h)$$

Recall that $G(h)$ is defined on a subset of all Hermitian metrics and thus it is more nonlinear*. In fact, it is more nonlinear in order to be able to use in the K-E geometry. This observation was made by Tian.

(*: This was observed by Tian)

We need to elaborate the notations. $h_E = (h - h^{-1})^{n+1}$ is the virtual Hermitian metric on the virtual bundle $(K_X^{-1} - K_X)^{n+1}$

Since $E = \sum_{k=0}^{n+1} C_{n+1}^k (-1)^k K_X^{n+1-2k}$, So

$$R(h_E) = \sum_{k=0}^{n+1} C_{n+1}^k (-1)^k \ (-(n+1)-2k) R(h)$$

Thus the Euler-Lagrange equation is given by

$$\int_X (-1)^k C_{n+1}^k \ (n+1-2k)^{n+1} u h^{-1} \omega^n - \lambda \int_X u = 0$$

where λ is the Lagrange mutiplier, this is because the restriction is

$$\int_X u = 0.$$

Thus

$$h^{-1} \omega^n = \text{constant}$$

But since

$$\int_n h = \int_X \omega^n$$

we have

$$\omega^n = h$$

$$Ric(\omega) = -\partial\bar\partial \log \omega^n = -\partial\bar\partial \log h = \omega.$$

K-E.

The functional G on K can be expressed explicitly as follows. Let g_0 be a reference Kähler metric. Then for any $g \in K$, we have

$$\omega_g = \omega_{g_0} + \partial\bar\partial\varphi$$

The normalization condition is

$$\int_X e^{-\varphi} h_0 = \int_X \omega_{g_0}^n = \int_X c_1(X)^n$$

Define a family of Kähler metrics g_t by

$$\omega_{g_t} = \omega_{g_0} + \partial\bar\partial(t\varphi)$$

Then

$$G(g) = a_n\left(\int_0^1 \int_X (\varphi(\omega_{g_0}^n - \omega_{g_t}^n))\, dt - \int_X \varphi\, \omega_{g_0}^n\right)$$

$$= a_n\left(J_{g_0}(\varphi) - \int_X \varphi\, \omega_{g_0}^n\right)$$

where $J(\varphi)$ is the Aubin's J-functional.

Example 2. Let $L \to X$ be an ample line bundle over X. Let h_0, h_1 be two ~~vecto~~ Hermitian metrics on L. Let

$$A = \prod_i \frac{x_i}{1-e^{-x_i}} \qquad \text{Todd polynomial}$$

$$Q = \sum_k \sum_i x_i^k$$

We define

$$K(h) = D\left((AQ)_{n+1}\ (g_0, (h_0 - h_0^{-1})^n),\ (g_1, (h_1 - h_1^{-1})^n)\right)$$

We wish to express $K(h)$ interms of h_0, h_1. Since by Donaldson's lemma, the functional is independent of the choice of the ~~so~~ path connecting the two virtual metrics, we choose the path

$$g_0, (h_t - h_t^{-1})^n \quad \text{and then}$$
$$g_t \quad (h_1 - h_1^{-1})^n$$

In this way, up to $\mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial$, we have

$$BC\left((AQ)_{n+1}, \left(g_0, (h_0 - h_0^{-1})^n\right), \left(g_1, (h_1 - h_1^{-1})^n\right)\right)$$

$$= \sum_{i=0}^{n+1} BC(A_i, g_0, g_1)\, \cancel{Q_{n+1-i}(R(h_1, h_1))}\, Q_{n+1-i}\left(R\left((h_1 - h_1^{-1})^n\right)\right)$$

$$+ \sum_{i=0}^{n+1} A_i(R(g_0))\, BC\left(Q_{n+1-i}, (h_0 - h_0^{-1})^n, (h_1 - h_1^{-1})^n\right)$$

For $BC(Q_{n+1-i}, \cdots)$, we have

$$BC\left(Q_{n+1-i}, (h_0 - h_0^{-1})^n, (h_1 - h_1^{-1})^n\right)$$

$$= \sum_{j=0}^{n} (-1)^j C_n^j\, BC\left(Q_{n+1-i}, h_0^{n-2j}, h_1^{n-2j}\right)$$

$$= \sum_{j=0}^{n} (-1)^j C_n^j\, (n-2j)^{n+1-i}\, BC(B_{n+1-i}, h_0, h_1)$$

$$= \begin{cases} 0 & i \neq 1 \\ 2^n n!\, BC(Q_n, h_0, h_1) \end{cases}$$

On the other hand

$$Q_{n+1-i}\left(R\left((h_1 - h_1^{-1})^n\right)\right)$$

$$= \begin{cases} 0 & i \neq 1 \\ 2^n n!\, R(h_1)^n & i = 1 \end{cases}$$

Therefore, we have

$$BC\left((AQ)_{n+1},\ (h_0 - h_0^{-1})^n,\ (h_1 - h_1^{-1})^n\right)$$
$$= BC(A_1, g_0, g_1)\, 2^n n!\, R(h_1)^n + A_1(R(g_0))\, 2^n n!\, (Q_n, h_0, h_1$$

We have

$$BC(A_1, R(h_0), R(h_1))$$
$$= -\frac{1}{2} \log \frac{\det g_1}{\det g_0}$$

$$BC(Q_n, h_0, h_1)$$
$$= n\varphi \int_0^1 R(h_t)^{n-1}\, dt$$

Therefore

$$F(\varphi) = K(h) = 2^n n! \left(-\frac{1}{2} \int_X \log \frac{\det(R(e^{-\varphi} h_0))}{\det(R(h_0))} \right.$$
$$\cdot R(h_1)^n + n \int_X \varphi\, Ric(R(h_0)) \wedge \int_0^1 R(e^{-t\varphi} h_0)^{n-1}\, dt$$

The k - energy is defined to be

$$\gamma(\omega_0, \omega_1) = - \int_X \int_0^1 \varphi \left(\mathrm{Ric}(\omega_s) - \omega_s \right) \wedge \omega_s^{n-1}$$

where

$$\omega_1 = \omega_0 + \partial\bar\partial\varphi \quad , \qquad \omega_s = \omega_0 + s\,\partial\bar\partial\varphi$$

Example 2. Let

$$E = (n+1)(K^{-1}-K) \otimes (L-L^{-1})^n - n\,(L-L^{-1})^{n+1}$$

where L is the polarization of X with $c_1(L) = c_1(X)$

$K = K_X$ is the canonical line bundle. Let

$$\mathcal{D}(Ch_{n+1}, h_0, h_1) = \gamma(\omega_0, \omega_1)$$

where $\quad \omega_i = -\partial\bar\partial \log h_i$

Now assume that ~~V~~ ~~Φ~~ is a holomorphic vector field on X. Let $\omega_0 \in C_1(X)$. Let $\omega_t = \sigma_t^*(\omega_0)$ where σ_t is the flow on X defined by X (or $\mathrm{Re}\,X$, to be precise) Then we see that

$$\mathcal{D}(\varphi_1 \cdots \varphi_J, h_0, h_t) = \mathcal{D}(\varphi_1, \cdots, \varphi_J, \omega_0, \omega_t)$$

is $k \cdot t + l$. The constant k is the Futaki invariant. In particular, The Futaki invariant is indepent of the choice of the representitive of the cohomological class. If the Donaldson's functional has a lower bound, then the Futaki invariants is automatically zero.

Remark: The above discussions are also true in the case of extremal metrics.

Generalized Futaki invariant.

Let (X, ω) be a compact kähler manifold and let $E \to X$ be a virtual holomorphic bundle. Let G be a subgroup of $\mathrm{Aut}(X)$, the holomorphic automorphism group. Assume that G can be lifted to an automorphism of E preserving the fibers. Assume that X preserves the cohomological class $[\omega]$. Then

Prop:

$$\int_X \tilde{\varphi}\,(\Theta_h(X^*),\, R(h), \cdots, R(h)) \wedge \omega^{n-k+1}$$

$$- \frac{n-k+1}{k} \int_X f_X\, \tilde{\varphi}\,(R(h), \cdots R(h)) \wedge \omega^{n-k}$$

is independent of the choice of h. where f_X is the Hamiltonian function of X w.r.t. ω : $i_X\omega = \bar\partial f_X$

Prop. Let $\varphi = c_1^{m+1}$, h be the metric of the line bundle $L \to X$. Then the above is the Futaki invariant.

$$\pi(Y) = \frac{\omega(Y, x)}{\omega(x, x)}$$

$$\eta = \varphi\left(\mathcal{O}_n(x^*) + R(h)\right) \wedge \frac{\lambda}{H\partial\lambda}$$

$$\cancel{i(x)\,\eta = \bar{\partial}\eta} + \varphi\left(\mathcal{O}_n(x^*) + R(h)\right)$$

$$\left(i(x) - \bar{\partial}\right)\eta = \varphi\left(\mathcal{O}_n(x^*) + R(h)\right).$$