Calibrations III

Exceptional Geom's

$\mathbb{R}$ = real nos. $\mathbb{C}$ = complex nos $\mathbb{H}$ = quaternions $\mathbb{O}$ = octonions

In all cases: $\langle \cdot , \cdot \rangle$ and $\overline{(\ )} : F \to F$ involution.

(1) $|xy| = |x||y|$

(2) $\langle zx, y \rangle = \langle x, \bar{z} y \rangle$

(3) $\overline{xy} = \bar{y}\bar{x}$ and $|x|^2 = x\bar{x} = \bar{x}x$

$\mathbb{H}$ is not commutative
$\mathbb{O}$ " " associative.

Theorem (Artin) The subalgebra with 1: $\langle 1, x, y \rangle$, generated by any two elements $x, y \in \mathbb{O}$ is associative. (in fact contained in a quaternion subalgebra of $\mathbb{O}$).

Moufang Identities:

$$(xyx)z = x(y(xz))$$
$$z(xyx) = ((zx)y)x$$
$$(xy)(zx) = x(yz)x$$

Cayley-Dickson Process

A = an algebra with $\overline{(\ )}$.

Define mult. on $A \oplus A$ by:

$$(a,b)\cdot(c,d) \equiv (ac - \bar{d}b,\ da + b\bar{c})$$
$$\overline{(a,b)} \equiv (\bar{a}, -b)$$

This generates recursively

$$\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O}$$

Cross Products on $\mathbb{O}$

$$x \times y \equiv -\frac{1}{2}(\bar{x}y - \bar{y}x)$$

$$x \times y \times z \equiv \frac{1}{2}\{x(\bar{y}z) - z(\bar{y}x)\}$$

or alt of this

$$x \times y \times z \times w \equiv \frac{1}{4}\{\bar{x}(y \times z \times w)^* + \bar{y}(z \times x \times w) + \bar{z}(x \times y \times w) + \bar{w}(y \times x \times z)\}$$

- Each form is alternating (skew)
- ☆ $|x \times y| = |x \wedge y|$, $|x \times y \times z| = |x \wedge y \wedge z|$, $|x \times y \times z \times w| = |x \wedge y \wedge z \wedge w|$.

Idea In each case Re( ) is a calibration

Re( ) $= \pm 1 \iff$ Im(x) $\equiv \cancel{x}' = 0$

the eqns

$$\mathrm{Re}(x \times y) = 0$$

$$\mathrm{Re}(x \times y \times z) = \langle x', y'z' \rangle \qquad \text{for } x y z \in \mathrm{Im}\,\mathbb{O}$$

$$\mathrm{Re}(x \times y \times z \times w) = \langle x, y \times z \times w \rangle$$

$x, y, z, w \in \mathrm{Im}\,\mathbb{O}$

$$\mathrm{Im}(x \times y) = \frac{1}{2}[x, y] \quad \text{commutator.}$$

$$\mathrm{Im}(x \times y \times z) = \frac{1}{2}[x, y, z] \quad \text{associator}$$

$$\mathrm{Im}(x \times y \times z \times w) = \frac{1}{2}[x, y, z, w] \quad \text{coassociator.}$$

$$[x,y] \equiv xy - yx$$

$$[x,y,z] = (xy)z - x(yz)$$

$$-\tfrac{1}{2}[x\,y\,z\,w] = \langle y, zw\rangle x + \langle z, xw\rangle y$$
$$\langle x, yw\rangle z + \langle y, xz\rangle w$$

$\boxed{\text{Associative Geometry}}$

$\mathbb{R}^7 = \operatorname{Im} \mathbb{O}$

Def $\quad \varphi(x, y, z) = \langle x, y \cdot z\rangle$

associative calibration

φ is alternating

$|\varphi(e_1\, e_2\, e_3)| \leq |\langle e_1, e_2 \cdot e_3\rangle| \leq |e_1||e_2 e_3| = 1$

"=" $\iff$ $e_1 = e_2 e_3$ (+ permutation)

$\iff$ span$\{1, e_1\, e_2\, e_3\}$ is a $\mathbb{H}$-subalgebra

$$\boxed{|\varphi(x,y,z)| + \tfrac{1}{4}|[x,y,z]|^2 = |x \wedge y \wedge z|^2}$$

$\therefore$

$$|\varphi(x, y, z)| = |x \wedge y \wedge z| \iff [x, y, z] = 0$$

$\mathcal{G}(\varphi) =$ the set of can. or'd Im-parts of $\mathbb{H}$-subalg's of $\mathrm{Im}\,\mathbb{O}$.

$=$ 3-planes on which $[\cdot, \cdot, \cdot] = 0$

The associative 3-planes

$$G_2 \equiv \mathrm{Aut}(\mathbb{O})$$

$$G_2 = \{ g \in O_7 : g^* \varphi = \varphi \}$$ see def of φ.

$$\mathcal{G}(\varphi) \cong G_2 / SO_4$$

$SO_4 = Sp_1 \times Sp_1 / \mathbb{Z}_2$ acts on $\mathrm{Im}\,\mathbb{O} = \mathrm{Im}\,\mathbb{H} \oplus \mathbb{H}$ by $T_{(p,q)}(u, v) = (q u \bar{q}, p v \bar{q})$.

Associative submanifolds:

$M^3 \subset \mathbb{R}^7 = \mathrm{Im}\,\mathbb{O}$ or'd

$T_x M^3 \in \mathcal{G}(\varphi) \quad \forall x \in M$

<u>Diff Eq$^{\underline{n}}$</u> Express M (locally)

as a graph of

$$f : \Omega \to \mathbb{H}$$

$\Omega^{\text{open}} \subset \operatorname{Im}\mathbb{H}$ and $\operatorname{Im}\mathbb{O} = \operatorname{Im}\mathbb{H} \oplus \mathbb{H}$

<u>Thm</u> $gr(f)$ is associative $\iff$

$$\boxed{Df = \sigma f}$$

where Dirac

$$Df = -\frac{\partial f}{\partial x_1} i - \frac{\partial f}{\partial x_2} j - \frac{\partial f}{\partial x_3} k .$$

$$\sigma f = \frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_2} \times \frac{\partial f}{\partial x_3}$$

Pf Suffices to consider f linear

$$Df = -f(i)i - f(j)j - f(k)k$$

$$\sigma f = f(i) \times f(j) \times f(k)$$

Calculation shows: $x = i + f(i)$, $y = j + f(j)$ $z = k + f(k)$

$$\tfrac{1}{2}[x,y,z] = \mathrm{Im}(x \times y \times z)$$

$$= (T, \sigma(f) - D(f)) \in \mathrm{Im}\,\mathbb{H} \oplus \mathbb{H}$$

$$= \mathrm{Im}\,\mathbb{O}$$

(1) $gr(f)$ is assoc.

$\iff$ $[x,y,z] = 0$

$\implies$ $\sigma(f) = D(f)$

(2) $\sigma(f) = D(f) \implies [x,y,z] \in \mathrm{Im}\,\mathbb{H}$

But $[x,y,z] \perp x \perp y$ and $\perp z$.

$\mathrm{Im}\,\mathbb{H}$ comp's of $x\,y\,z$ are $i\,j\,k$

$\therefore$ $[x,y,z] = 0$

and $gr(f)$ is assoc.

qed (up to orientation)

Existence:

Thm $B^2 \subset \mathrm{Im}\,\mathbb{O}$ any C^ω submanifold. Then $\exists!$ (germ) associative 3-man. $M^3 \supset B$.

Pf C-K

Coassociative Geom

Def $\psi \in \Lambda^4 \mathrm{Im}\,\mathbb{O}$

$$\psi(x\,y\,z\,w) \equiv \tfrac{1}{2}\langle x, [y,z,w]\rangle$$

coassociative calibration

We have

$$\boxed{\psi(x,y,z,w)^2 + \tfrac{1}{4}\left|[x,y,z,w]\right|^2 = |x\wedge y\wedge z\wedge w|^2}$$

$\therefore$ ψ is a calibration and

$\psi(xyzw) = 1 \iff [x,y,z,w] = 0$

Calculation $\psi = *\varphi$

$$\therefore \quad \mathcal{G}(\psi) \xrightarrow{\approx} \mathcal{G}(\varphi)$$

$$\xi \longmapsto *\xi = \text{"}\xi^{\perp}\text{"}$$

Coassociative 4-planes.

Prop ξ = 4-plane in $\mathrm{Im}\,\mathbb{O}$

$\pm\xi$ is coassociative $\iff$ $x \cdot y \perp \xi$ $\forall\, x, y \in \xi$.

The equation

$$F : \Omega \to \mathrm{Im}\,\mathbb{H}$$

$$\Omega^{\text{open}} \subset \mathbb{H}$$

Write

$$F = F_1 i + F_2 j + F_3 k$$

F_α are $\mathbb{R}$-values

Consider:

$$\hat{D}F = -\nabla F_1 \cdot i - \nabla F_2 \cdot j - \nabla F_3 \cdot k$$

$$\tilde{\sigma} F = \nabla F_1 \times \nabla F_2 \times \nabla F_3$$

where ∇F_α is considered in $\mathbb{H}$

<u>Thm</u> $gr(F)$ is coassociative

$$\iff \boxed{\hat{D}F = \hat{\sigma} F} \quad *$$

<u>Cor</u> C^1 sol^{ns} of $*$ are C^ω.

Interesting Example Fix unit vector $u \in \mathrm{Im}\,\mathbb{H}$

$$H : S^3 \longrightarrow S^2$$

$$H(q) = \frac{\sqrt{5}}{2}\,\bar{q}\,u\,q. \qquad q \in \mathbb{H},\ |q| = 1$$

<u>Thm</u> The cone on $gr(H)$ is co-associative.

i.e. The graph of $h : \mathbb{H} \to \mathrm{Im}(\mathbb{H})$

$$h(q) = \frac{\sqrt{5}}{3} \frac{1}{\|q\|} \bar{q} u q.$$

is absolutely vol. min.

(Lipschitz but not C^1)

Osserman.

Cayley Geometry

$$\Phi \in \Lambda^4 \mathbb{O}^*$$

Def $\Phi(x,y,z,w) = \langle x, y \times z \times w \rangle$

Cayley 4-form.

Thm $\Phi(x \wedge y \wedge z \wedge w)^2 + |\mathrm{Im}(x \times y \times z \times w)|^2 = |x \wedge y \wedge z \wedge w|^2$

Cor Φ is a <u>calibration</u> and $\pm \xi = x \wedge y \wedge z \wedge w$ is a Φ-plane $\iff \mathrm{Im}(x \times y \times z \times w) = 0$.

$$G(\Phi) = \text{"the Cayley 4-planes"}$$
$$\cong Spin(7)/K$$
$$K = SU_2 \times SU_2 \times SU_2 / \mathbb{Z}_2$$

Fact $\boxed{\Phi = 1 \wedge \varphi + \psi.}$

Other characterizations (of Cayley 4-planes)

$\pm\xi \in G_4(\mathbb{O})$ is a Cayley 4-plane

$\iff$ $x \times y \times z \in \xi \quad \forall\, x, y, z \in \xi$

$\iff$ ξ is $\mathbb{C}$ wrt every $\mathbb{C}$-structure $J_{x \times y}$ on $\mathbb{O}$ for all $x, y \in \xi$

$\iff$ $\omega|_\xi$ is antiselfdual

where $\omega \equiv$ Kahler form for J_e

$\mathbb{O} \equiv \mathbb{H} \oplus \mathbb{H} \cdot e$

Note $\quad \Phi = -\frac{1}{2}\omega \wedge \omega + \mathrm{Re}\{dz\}$ for J_e.

The Cayley Eq[1]

$\Omega^{\text{open}} \subset \mathbb{H}$

$f : \Omega \to \mathbb{H}$

Define 3 operators $\qquad e_1, \ldots, e_4$ any ~~or.~~ or o.n. basis of $\mathbb{H}$

1. Dirac Operator

$$Df \equiv \sum_{j=1}^{4} (\nabla_{e_j} f)\, \bar{e}_j$$
$$= \frac{\partial f}{\partial x_0} - \frac{\partial f}{\partial x_1} \cdot i - \frac{\partial f}{\partial x_2} j - \frac{\partial f}{\partial x_3} \cdot k$$

$$\left(\text{also} \quad -\bar{D} f = \frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_1} i + \frac{\partial f}{\partial x_2} j + \frac{\partial f}{\partial x_3} k \right)$$
$$D\bar{D} = \bar{D} D = +\Delta$$

2. $$\sigma f \equiv \sum_{j_1 < j_2 < j_3} \{ (\nabla_{e_{j_1}} f) \times (\nabla_{e_{j_2}} f) \times (\nabla_{e_{j_3}} f) \} \overline{(e_{j_1} \times e_{j_2} \times e_{j_3})}$$

3. $$\delta f \equiv \sum_{j_1 < j_2} \{ (\nabla_{e_{j_1}} f) \times (\nabla_{e_{j_2}} f) \} \overline{(e_{j_1} \times e_{j_2})}$$

Thm Graph(f) is Cayley $\iff$ $Df = \sigma f$ and $\delta f = 0$

If $\det(df) \neq 1$, $\delta f = 0$ automatically

Special Case: Write

$$f = +\bar{D}g. \qquad g:\Omega \to \mathbb{H}$$

Then $\boxed{\Delta g = \sigma(\bar{D}g)}$ is The Cayley equation

Implicit F$^{\underline{n}}$ Techniques Apply.

Globalize to Manifolds

GM 1

$G \subset O_n$ Lie-subgroup

Def A <u>G-manifold</u> is a riemannian n-manifold with holonomy group $\subseteq G$. (conj. class)

SU_m-man.'s	G_2-man.'s	$Spin_7$-manifolds
(Ricci-flat Kahler "Calabi-Yau")	$(n=7)$	$(n=8)$

G_2-man.'s $\swarrow$ Assoc. sub.'s, $\searrow$ Coassoc sub.'s

Slag sub's | Assoc. sub.'s | Coassoc sub.'s | Cayley sub.'s.

I) SU_m-holon. on $\mathbb{C}$-man.

ω = Kahler form

Φ = parallel $(0,n)$-form.

$$\varphi \equiv \operatorname{Re} \Phi$$

has comass $\equiv 1$ and $d\varphi = 0$ $(\nabla\varphi = 0)$.

II) $\varphi \in \mathcal{E}^3$ $\qquad\qquad \psi = *\varphi$

parallel assoc. cal. $\qquad\qquad$ parallel assoc cal.

III) $\Phi \in \mathcal{E}^4$ parallel

CM2

Remember $\varphi \in \Lambda^p T_x X$ is fixed by holon. G_x

$\iff$ φ has a parallel extension to X.

(In G_2, $Spin_7$-cases, φ, Φ determine the Riem. structure and $d_{*\varphi}\varphi = d\varphi = 0 \iff$ - -)

$$\boxed{\text{Deformations of compact } \varphi\text{-submanifolds}}$$

Slag. Case $M^m \subset X$ cpt Slag in SU_m-manifold.

Theorem: The moduli space of all S.L. subman's in a C^1-nbhd of M is a C^ω man. of dimension $b_1(M)$. It's tang space is naturally identified with $\mathbb{H}^1_M$ = harm 1-forms.

Deformations of sp. Lagrangian

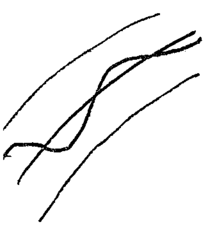
$M \subset X$ S-Lag., cpt oriented.

$T^*M \cong TM \overset{J}{\cong} NM \cong N^*M.$

① Fix nbhd $\mathcal{U}$ of $M \subset N(M)$ s.t.

$$\exp : \mathcal{U} \longrightarrow X$$

is a diffeo. onto an open set $\hat{\mathcal{U}} \subset X$.



② Any section $V \in \Gamma(N)$ gives an embedded sub. $M_V = \exp(V) \subset \mathcal{U}$, and every sub. in a C^1-nbhd of M $\mathcal{U}$ is of this form.

③ M_V is S-Lag $\iff \omega|_{M_V} = \varphi|_{M_V} = 0$

$\iff \exp(V)^*\omega = \exp(V)^*\psi = 0$

Consider the map

$$\mathcal{E}^1(M) = \Gamma(N) \xrightarrow{\Psi} \underline{\Omega}^2(M) \oplus \mathcal{E}^n(M)$$

$$V \longmapsto (\exp(V)^*\omega, \exp(V)^*\psi)$$

⊛ $\Psi^{-1}(0) \cap \mathcal{U} \equiv$ all S-Lag submans in $\mathcal{U}$.

GM 4

We now compute $d\Psi|_0$.

Fix $V \in \Gamma(N)$ and $x \in M$.

Choose o.n. frame field
$e_1\ Je_1 \ \cdots\ e_n\ Je_n$ near x

s.t.
(i) Along M $e_1, \ldots, e_n$ are tangent to M
($Je_1 \ldots Je_n$ " normal ..)

(ii) $(\nabla_{e_i} e_j)^T_x = 0$

Let $\omega_1\ J\omega_1 \ \cdots\ \omega_n\ J\omega_n$ be dual coframe.

((i) $\Rightarrow$ $\omega_1 \ldots \omega_n|_M$ dual to $e_1 \ldots e_n$
$J\omega_1 \ldots J\omega_n|_M = 0$.)

$$\left[\begin{array}{l} \omega = \sum_{j=1}^{n} \omega_j \wedge J\omega_j \\ \psi = \operatorname{Im}(\omega_1 + iJ\omega_1)\wedge \cdots \wedge (\omega_n + iJ\omega_n) \end{array}\right.$$

We want to compute:

$$\frac{d}{dt} \exp(tV)^*(\omega)\Big|_{t=0}$$

$$\frac{d}{dt} \exp(tV)^*(\Psi)\Big|_{t=0}$$

$$V = \sum V_k J e_k \simeq \sum V_k e_k \simeq \sum V_k \omega_k.$$

$$\begin{cases} \exp(tV)^*(\omega_k) = \omega_k + O(t) \\ \exp(tV)^*(J\omega_k) = (dV_k)t + O(t^2) \quad \underline{\underline{\text{at } x}} \end{cases}$$

$$dV_k = \sum_{j=1}^{n} V_{kj}\,\omega_j \qquad V_{kj} = e_j \cdot V_k$$

$$\exp(tV)^*(\omega) = \sum_{k=1}^{n} \omega_k \wedge \Big(\sum_{j=1}^{n} V_{kj}\,\omega_j\Big) \cdot t + O(t^2)$$

$$= t\,dV + t^2$$

$$\Big(dV = \sum \omega_j \wedge \nabla_{e_j} V = \sum \omega_j \wedge \nabla_{e_j} \sum V_k \omega_k$$
$$= \sum \omega_j\, V_{kj}\, \omega_k = \sum V_{kj}\, \omega_j \wedge \omega_k \Big)$$

$$\exp(tV)^*(\psi) = \mathrm{Im}\Big\{ \prod_{k=1}^{n} \Big(\omega_k + it\sum_{j=1}^{n} V_{kj}\,\omega_j\Big)\Big\} + \cdots$$

$$= \mathrm{Im}\{(\omega_1 + it\sum V_{1j}\,\omega_j) \cdots (\omega_n + it\sum V_{nj}\,\omega_j)\}$$

$$= t\Big(\sum_{j=1}^{n} V_{jj}\Big)\,\omega_1 \wedge \cdots \wedge \omega_n + O(t^2)$$

$$= t * (d^*V) + O(t^2)$$

$$* d^* V = d(*V)$$
$$= \sum_{j=1}^{n} \omega_j \wedge \nabla_{e_j} \left(\sum_{k=1}^{n} V_k (-1)^{k-1} \omega_1 \wedge \dots \wedge \hat{\omega}_k \wedge \dots \wedge \omega_n \right)$$
$$= \sum_{j=1}^{n} V_{jj} \, (\omega_1 \dots \omega_n).$$

Prop

$$\begin{array}{ccc} \Gamma(N) & \xrightarrow{d\Psi_0} & \mathcal{E}^2(M) \oplus \mathcal{E}^n(M) \\ \cong & & \| \\ \mathcal{E}^1(M) & \xrightarrow[(-d,\, *d^*)]{} & \mathcal{E}^2(M) \oplus \mathcal{E}^n(M) \end{array}$$

$$\boxed{\ker(d\Psi_0) = \mathbb{H}^1 = \text{harm } 1\text{-forms.}}$$

Now Use the Implicit F^n Thm in Banach Spaces.

(I) $\Psi : \mathcal{E}^1(M) \longrightarrow d\mathcal{E}^1(M) \oplus d\mathcal{E}^{n-1}(M)$

i.e. $\exp(V)^*(\alpha)$ and $\exp(V)^*(\psi)$ are <u>exact</u>.

Pf $\exp(V)$ is homotopic to $j : M \subset X$ and $j^*\omega = j^*\psi = 0$.

$\therefore [\exp(v)^*\omega] = [j^*\omega] = 0$

$[\quad — \quad \psi] = [j^*\psi] = 0$

in $H_{deR}(M)$ $\boxtimes$

Now

$$\mathcal{E}^1(M)_{C^{1,\alpha}} \xrightarrow{\Psi} d\mathcal{E}^1(M)_{C^{0,\alpha}} \oplus d\mathcal{E}^{n-1}(M)_{C^{0,\alpha}}$$

a bounded map of Banach manifolds

i.e. $$\mathcal{E}^1(M)_{C^{1,\alpha}} \xrightarrow{\Psi} \mathcal{E}^2(M)^{exact}_{C^{0,\alpha}} \oplus \mathcal{E}^n(M)^{exact}_{C^{0,\alpha}}.$$

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and $d\Psi_0 = (d, *d^*)$

is an isomorphism

$\therefore$ IFThm $\Rightarrow$ $\Psi^{-1}(0)$ is a submanifold of dim $=$ dim $H^1(M;\mathbb{R})$

Similar Result
in Coassoc. Case.

Tang space $\cong$ Anti-self dual
Harm 2-forms.

<u>Assoc. Case</u>

$\exists$ Right Cliff Mult by tang vectors
on normal bundle
$\therefore$ $\exists$ Dirac op. $\not{D}$

Tang space $\simeq$ ker $\not{D}$.

Cayley is similar

Here we do not know
about mod. space

Spinors and Calibrations

For simplicity $V = \mathbb{R}^{8k}$ with $\langle\cdot,\cdot\rangle$ = standard

$$Cl(V) \equiv \text{Cliff alg of } (V, \langle\cdot,\cdot\rangle)$$
$$= \{1, e_1, \ldots e_{8k} : e_i e_j + e_j e_i = -2\delta_{ij} \mathbb{1}\}$$
$$\cong \Lambda^* V \text{ as } \underline{v\text{-spaces}}.$$

Basic Fact $\quad Cl(V) = \operatorname{End}_{\mathbb{R}}(S)$

where $\quad S \cong \mathbb{R}^{16^k}$ is the (real) $\underline{\text{spinor}}$ $\underline{\text{space}}$

I. $\quad \exists$ inner prod. $\langle\cdot,\cdot\rangle_S$ on S

$$\langle e\sigma_1, e\sigma_2\rangle_S = \langle\sigma_1, \sigma_2\rangle \quad \forall e \in V \ |e|=1$$

II $\quad \exists$ inner prod $\langle\cdot,\cdot\rangle_{Cl}$ on Cl

$$\langle A, B\rangle_{Cl} = \operatorname{tr}_S A^t \cdot B$$

III $\quad \exists$ inner prod $\langle\cdot,\cdot\rangle_\Lambda$ on Λ^* (standard L^2)

$\underline{\text{Lemma}}$ $\quad \langle A, B\rangle_{Cl} = 16^k \langle A, B\rangle_\Lambda$

$\underline{\text{Pf}}$ $\quad \langle A, B\rangle_{Cl} = \sum_{\alpha=1}^{16^k} \langle A(\sigma_\alpha), B(\sigma_\alpha)\rangle_S$

$$\langle e_{i_1} \cdots e_{i_p}, e_{i_1} \cdots e_{i_p}\rangle_{Cl} = \sum_\alpha \cdots$$

any unit vector.

$$= \sum_\alpha \langle\sigma_\alpha \sigma_\alpha\rangle = 16^k$$

Main Principle The square of a spinor yields interesting calibrations.

Given $\sigma \in S$, $(|\sigma| = 1)$ define $\sigma\circ\sigma \in Cl$ by

$$\boxed{\sigma\circ\sigma(\tau) \equiv \langle \tau, \sigma\rangle \sigma}$$

Lemma Given $\varphi \in Cl$

$$\boxed{\langle\varphi, \sigma\circ\sigma\rangle_{Cl} = \langle \sigma, \varphi\cdot\sigma\rangle_S}$$

Pf $\langle\varphi, \sigma\circ\sigma\rangle_{Cl} = tr(\varphi^t\circ(\sigma\circ\sigma))$

$$= \sum_\alpha \langle \varphi(\sigma_\alpha), (\sigma\circ\sigma)(\sigma_\alpha)\rangle_S$$

Choose $\sigma_1 \cdots \sigma_{16^k}$; $\sigma_1 = \sigma$ $\quad = \langle \varphi(\sigma), \sigma\rangle_S$ qed.

Now under $Cl(V) \cong \Lambda^* V$ decompose

$$N\, \sigma\circ\sigma \equiv \sum_{p=0}^{n} \psi_p \qquad \begin{array}{l} n = 8k \\ N = 16^k \end{array}$$

Thm Each ψ_p is a calibration

Pf $\xi \in \Lambda^p V$ a unit simple vector

$$\langle \xi, \sigma\circ\sigma\rangle_{C\ell} = \langle \sigma, \xi\sigma\rangle \le |\sigma||\xi\sigma| = 1$$

$$\| $$

$$N\langle \xi, \sigma\circ\sigma\rangle_\Lambda$$

$$\| $$

$$\langle \xi, N\sigma\circ\sigma\rangle_\Lambda$$

qed.

Can see many things:

① $\mathcal{G}(\varphi_p) = \{\xi : \xi\sigma = \sigma\}$.

② $Spin_n \xrightarrow{\mathbb{Z}_2} SO_n$ acts on S

$G_\sigma = \{g \in Spin_n : g\sigma = \sigma\}$.

Then G_σ acts on $\mathcal{G}(\varphi_p)$ $\forall p$.

Ex $k=1$ $\quad S = S^+ \oplus S^-$

3 reps. of Spin 8: $\quad V \quad S^+ \quad S^-$ all $\mathbb{R}^8$

Triality etc

Spin$_8$ is transitive on unit vectors in each.

Choose $\sigma \in S^+$ $\quad |\sigma| = 1$

$$\boxed{16\,\sigma \circ \sigma = 1 + \Phi + *1}$$

Cayley calibration

Spin$_7$ - fixes σ

$\therefore$ acts on $G(\Phi)$.

So on manifolds

Parallel spinors square to give ~~interesting~~ calibrations.