Lecture four Stability of algebraic manifolds

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A line bundle L over a compact complex manifold X is ample, if there is a Hermitian metric h on L such that

$$-\sqrt{-1}\,\partial\bar{\partial}\log h > 0$$

A line bundle L over a compact complex manifold X is very ample, if $\dim H^0(X, L) > 1$ and if we choose a basis $S_0, \cdots, S_{d-1}$ of $H^0(X, L)$ such that the map

$$X \longrightarrow \mathbb{CP}^{d-1}, \quad x \mapsto [S_0(x), \cdots, S_{d-1}(x)]$$

is an embedding of X into $\mathbb{CP}^{d-1}$. If X is embedded into some $\mathbb{CP}^N$, then X is an algebraic manifold, and we say that L is very ample.

The following Theorem of Kodaira is classical:

Theorem: If L is an ample line bundle, then for some $k > 0$, L^k is very ample.

In order to prove the theorem, we can use the $\bar{\partial}$-estimates. We have the following result from ~~Damilly~~ Demailly.

Theorem Suppose (X, g) is a compact kähler manifold of complex dimension n. L is a line bundle on X with Hermitian metric h. Let ψ be a function on X. Assume that

$$< \partial\bar\partial\psi + Ric(h) + Ric(g),\ v \wedge \bar v > \geq C \|v\|_g^2$$

for any tangent vector v of type (1.0) pointwisely. Then for any C^∞ L-valued $(0,1)$-form η on X with

$$\bar\partial \eta = 0$$

and

$$\int_X \|\eta\|^2 e^{-\psi} dV_g < +\infty$$

There exists C^∞ section u of L such that

$$\bar\partial u = \eta$$

and

$$\int_X \|u\|^2 e^{-\psi} dV_g \leq \frac{1}{C} \int \|\eta\|^2 e^{-\psi} dV_g$$

where dV_g is the volume form of g and $\|\cdot\|$ is induced by both h and g.

One remark we would like to make here is that we allow ψ to have singularities. The typical choice of ψ would be $\psi = C(\log r)$. If C is large enough, the function $e^{-\psi}$ is not integrable, which will force u to have zero's up to certain order. In many case, it will well shape the solutions.

The Kodaira embedding theorem is implied by the following

Prop. Let $k \gg 0$. Then for any $x, y \in X$, and a neighborhood $\{U, \varphi_U\}$ of x, and any vector v, we can find a holomorphic section S such that

$$s(x) = 0, \quad S(y) \neq 0$$
$$~~dst~~ \; dS \circ \varphi_U^{-1} \big|_{\varphi_U(x)} = v$$

Proof of the Kodaira Theorem from the above proposition. ~~First~~ Let

$$\varphi: X \longrightarrow \mathbb{CP}^N, \quad x \longmapsto [S_0(x), \cdots, S_{d-1}(x)]$$

be the embedding. Then if $x \neq y$

$$(S_0(x), \cdots, S_{d-1}(x)) \neq k\,(S_0(y), \cdots, S_{d-1}(y))$$

Let $\quad S = \sum C_i S_i$. Then we have

$$0 = k\, S(y)$$

a contradiction. So φ is 1-1. On the other hand φ is an ~~em~~ immersion because essentially we can prescribiy all the first order derivatives.

$\square$.

In order to prove the proposition, we use Demailly's $\bar\partial$-estimate. For the sake of simplicity, we assume that $y \in U$ where U is a geodesic neighborhood of x. We notice that the size of U depends on the injectivity radius of the manifold X. So this proof is for a

Single manifold. not for a family of manifolds

On the neighborhood U, assume that L is trivial. Let $(z_1, \cdots z_n)$ be the local holomorphic coordinates. A section of L or L^k $(k \gg 0)$ is a holomophic function. At least on the neighborhood U, we can define a holomophic function f such that

$$f(x) = 0, \quad df(x) = v, \quad f(y) \neq 0$$

Using ~~parti~~ cut-off function, we can extend f to be a global C^∞ section of L^k (k to be determined). Let $\eta = \bar\partial f$. Let $\psi = C_1 \log(\sum |z_i|^2)$ where C is a constant to be determined. We choose C_1 large enough such that

$$\int_B \int e^{-\psi} \, dV < +\infty$$

And we then choose k large enough such that

$$\langle 2\partial\bar\partial\psi + k\,Ric(h) + Ric(g), v \wedge \bar v \rangle$$

$$\geq C \|v\|_g^2$$

By $\bar\partial$-estimate, we have u s.t. $\bar\partial u = \eta$ and

$$\int_X \|u\|^2 e^{-2\psi} dV_g \leq \frac{1}{C} \int \|\eta\|^2 e^{-2\psi} dV_g < +\infty$$

Since $\int e^{-\psi} = +\infty$, we have $u(x) = 0$, $du(x) = 0$. Thus ~~$\bar\partial(f-u) = 0$ but~~ Let $S = f - u$. Then S is holomophic and S satisfies the required condition.

A naturally question about the Kodaira's embedding theorem is that how large ~~we~~ k we shoud choose in order that L^k is very ample. From the above proof, we know that k depends on the injectivity radius and possibly the bound of the curvature etc. How ever, we have the following important

Fujita Conjecture : $(n+2)L + K_X$ is very ample if $L \to X$ is ample. $(n+1)L + K_X$ is free if L is ample.

Note: $L \to X$ is called free, if $\forall x \ \exists \ S \in H^0(X, L)$ such that $S(x) \neq 0$.

Demailly used the method of the solution of degenerated Monge-Ampere equation and proved that $12n^n L + 2K_X$ is very ample. After his work, many peope, Kollar, ~~etc~~ Ein-Lazarsfeld-Nakamaye ~~prov~~ improved his result. Recently, Siu proved the following result

Theorem $mL + K_X$ is very ample if

$$m > 2\left(n+2 + n\binom{3n+1}{n}\right)$$

$mL + K_X$ is free if $m = O(n^2)$.

Siu's method is basically ~~algebraic~~-geometric, which is very beautiful. At the risk of making it more confusing, I try to explain his proof in a naive way (Let me state one more time that Siu's method is very deep and beautiful).

Key step 1. Both Siu and Demailly used the Hilbert polynomial. Let F be a line bundle and $\mathcal{F}$ be a coheret sheaf. Then

$$\dim \sum (-1)^r \dim H^r(X, (mL+F)\otimes \mathcal{F})$$

is a polynomial of degree at most n in the variable m.

Since it is a polynomial. for m fixed but large

$$\dim H^0(X, (mL+F)\otimes \mathcal{F})$$

will be large. So we would have enough section to play with.

Key step 2. "Assume" that we are already find some m depending only on n such that other than a fixed point $x_0 \in X$, ~~the line bun~~ $\exists\, S \in H^0(X, L^m)$ such that $S(x) \neq 0 \ \forall x \neq x_0$. Then we can form a singular metric $\log (\sum |S_i(x)|^{\varepsilon})$ of L at x_0. Using this and the $\bar\partial$-estimate, we can find the required section.

Next we discuss the recenty result of Donaldson.

Theorem Let (X, L) be a polarized kähler manifold. Let $\mathrm{Aut}(X, L)$ be discrete. If X admits a Kähler metric of constant scalar curvature, then X is Hilbert-Mumford stable.

There are two main steps in Donaldson's proof.

Step 1. Donaldson proved that if $Aut(X, L)$ is discrete and if X admits a kähler metric of constant scalar curvature, then X is balanced.

Step 2. By the theorem of Luo and S. Zhang, balanced manifolds are Hilbert-Mumford stable.

Definition of stability. We assume that $X \subset \mathbb{CP}^N$ is already embedded into certain complex projective space. The Hilbert polynomial

$$P(m) = \dim H^0(X, L^m)$$

is a polynomial of m of degree n. Grothendieck proved the existence of a compact variety, called the Hilbert scheme with the fixed Hilbert polynomial, which parametrized all the subscheme of $\mathbb{CP}^N$ with the Hilbert polynomial. The Hilbert scheme is called $Hilb_h$. On $Hilb_h$, there is a canonical ample line bundle $\mathcal{L} \to Hilb_h$. Assume that $X \subset \mathbb{CP}^N$. Then $Aut(\mathbb{CP}^N)$ acting on $Hilb_h$ and has a linearization on the line bundle $\mathcal{L}$. We have the following ~~geometric critetia of~~ definition of stability:

A point $x \in H = Hilb_h$ is called stable with respect to $G = Aut(\mathbb{CP}^N)$, $\mathcal{L}$ and the given linearization, if x has finite stabilizer and for some $m \geq 1$, there exists a section $t \in \Gamma(Hilb_h, \mathcal{L}^m)^G$ such that

1. $H_t = H - V(t)$ is affine, where $V(t)$ is the zero locus of t.
2. $x \in H_t$ or in other terms, $t(x) \neq 0$,
3. G acting on H_t is a closed action.

Definition of "Balanced". Let (X, L) be a polarized Kähler manifold. Assume that $X \subset \mathbb{CP}^N$. If $S_0, \cdots S_N$ be the standard section of the hyperplane bundle $\xi \to \mathbb{CP}^N$ and if

$$\int_X \frac{\langle S_i, S_j \rangle}{\Sigma |S_i|^2} = \delta_{ij}$$

Then we say that X is balanced.

The concept "balanced" is equivalent to the following:

Proposition: If X is a Kähler manifold such that For $S_0, \cdots, S_{N-1} \in H^0(X, L^m)$ an orthonormal basis of the Hermitian vector space with respect to the L^2-inner product, the pointwise sum

$$\Sigma \|S_i\|^2 = \text{const}.$$

Then X is balanced.

A sketch of the proof that constant scalar curvature implies

$$\sum \|S_i\|^2 = \text{const}$$

First, if m is large enough, then we have the Tian-Yau-Zelditch expansion

$$\sum \|S_i\|^2 \sim m^n\left(a_0 + \frac{a_1}{m} + \frac{a_2}{m^2} + \cdots\right)$$

where $a_0 = 1/\text{vol}(X) = 1/c_1(X)^n$ and I proved that $a_1 =$ scalar curvature. Thus by our assumption, $a_1 = \text{const}$. Thus if m is large,

$$\sum \|S_i\|^2$$

is almost constant to some acorracy. Donaldson proved

1. After a change of the metric, one can make
$$\sum \|S_i\|^2 = \text{const} + O\left(\frac{1}{m^l}\right)$$
for any l;

2. After a change of the frame of $\mathbb{CP}^N$, one can make
$$\sum \|S_i\|^2 = \text{constant}.$$

A final remark:

We can prove that, formally, Chang of metrics
$\sum \|S_i\|^2 = \text{const}$,

But we can never prove the convergence. It was Donaldson's insight to make change of frame of $\mathbb{CP}^N$ and done the job, which involves quite a few analytic estimates.

END.

Thank you for attending my talks. I am very happy to discuss with you about the topics. My e-mail is zlu@math.uci.edu.