Tian: Kähler metrics and holomorphic foliations

(X.X. Chen & Tian)

- Geometric Motivations.

Let $[M, \Omega]$ be a polarized Kähler manifold (cpt) with Kähler class Ω.

$$K_\omega = \{\varphi \in C^\infty(M,\mathbb{R}) \mid \omega + \partial\bar\partial\varphi > 0\}, \quad [\omega] = \Omega$$
$$K_\Omega = \{\omega \text{ Kähler metrics} \mid [\omega] = \Omega\}$$

Calabi: ω is extremal Kähler if $\bar\partial S(\omega)$ induces a holomorphic v. field.

In particular, any Kähler metrics with constant scalar curv. are extremal.

Ex. If $c_1(M) = \lambda[\Omega]$, then any K-metric with constant scalar curv. is Kähler-Einstein (Hodge theory).

Fact. Any Kähler metric with constant scalar curv. is a critical point of the K-energy by T. Mabuchi

$$\nu_\omega(\varphi) = -\frac{1}{V}\int_0^1\int_M \dot\varphi_t(S(\varphi_t) - \mu)(\omega + \partial\bar\partial\varphi_t)^n, \quad n = \dim_{\mathbb{C}} M$$

where φ_t is a path from 0 to φ.

$$\nu_\omega(\varphi) = \frac{1}{V}\int_M \log\left(\frac{\omega_\varphi^n}{\omega^n}\right)\omega_\varphi^n + \underbrace{\text{lower order terms}}_{\text{bdd by } C^0\text{-norm of } \varphi}.$$

Theorem 1 If (M,Ω) admits a Kähler metric of constant scalar curvature, then $\nu_\omega(\varphi) \geq -C_\omega$ where $[\omega] = \Omega$. Moreover, if $S(\omega) = \mu$, then $C_\omega = 0$, i.e., the absolute minimum of ν_ω is attained by the Kähler metric of constant scalar curv.

— conjectured by Tian. In fact, one expects certain properness.

— essentially due to Bando - Mabuchi (also Tian) in case of Kähler - Einstein metrics.

— proved by X.X. Chen when $C_1(M) \leq 0$.

— There is an analogous statement for any extremal metrics if one replaces the K-energy by certain modified K-energy.

<u>Corollary</u>: If $(M, \underset{c_1(L)}{\overset{\shortparallel}{\Omega}})$ is a polarized manifold which admits a Kähler metric of constant scalar curv., then (M, Ω) is asymptotic semi-K-stable.

e.g. if $M \hookrightarrow \mathbb{CP}^N$ by $H^0(M, L^k)$ and $\{\sigma_t\} \subset SL(N+1)$ is a one-parameter subgroup, then

$$\lim_{t\to\infty} \nu_\omega'\left(\frac{1}{k}\sigma_t^*\omega_{FS}\Big|_M\right) \geq 0$$

in case $\sigma_t(M) \to M_\infty$ exists, $Re\left(F_{M_\infty}(\sigma_t'(0))\right) \geq 0$.

<u>Theorem 2</u>. For any given polarized (M, Ω), there is at most one extremal Kähler metric modulo holomorphic automorphisms.

— Kähler - Einstein : Calabi - Bando - Mabuchi (1986)

— $C_1(M) \leq 0$: X. X. Chen (1998)

— $\Omega = C_1(L)$ (algebraic) : S. Donaldson (2002).

"Outlined proof" (Mabuchi, Semmes, Donaldson, Chen)

$$\mathcal{K}_\Omega = \{\omega : \text{Kähler metrics} \mid [\omega] = \Omega\} = \mathcal{K}_\omega = \{\varphi \mid \omega + \partial\bar\partial\varphi > 0\}$$

Define

$$T_{\omega_\varphi}\mathcal{K}_\Omega = \{\psi \in C^\infty(M, \mathbb{R}) \mid \int_M \psi\,\omega_\varphi^n = 0\}$$

$$\|\psi\|_\varphi^2 = \frac{1}{V}\int_M \psi^2 \omega_\varphi^n$$

This gives a metric on $\mathcal{K}_\Omega$ (in fact, formally negatively curved)

Observation (Mabuchi): The K-energy is convex w.r.t. this L^2-metric on $\mathcal{K}_\Omega$.

In fact, if $\{\varphi_t\} \in \mathcal{K}_\omega$ is a geodesic path, then

$$\frac{d^2}{dt^2}\nu_\omega(\varphi_t) = \frac{1}{V}\int_M \Big|\underbrace{\nabla^{2,0}\dot\varphi_t}_{= D^{1,0}\partial\dot\varphi_t}\Big|^2_{\omega_{\varphi_t}} \omega_{\varphi_t}^n \geq 0$$

"=" holds iff $D^{1,0}\partial\dot\varphi_t \equiv 0$, i.e., $\bar\partial\dot\varphi_t$ induces holomorphic vector fields.

Geodesic equation: Write $\omega_{\varphi_t} = \omega + \partial\bar\partial\varphi_t = g_{\varphi_t\alpha\bar\beta}\,dz_\alpha \wedge d\bar z_\beta$.

$$\varphi''(t) - g_{\varphi_t}^{\alpha\bar\beta}\frac{\partial\varphi'}{\partial z_\alpha}\frac{\partial\varphi'}{\partial\bar z_\beta} = 0$$

Introduce $z_0 = t + is$, $s \in S^1$, $t \in [a,b]$, then the above equation can be rewritten as

$$(\omega + \partial\bar\partial\Phi)^{n+1} = 0$$

where $\Phi : \underbrace{\Sigma}_{= [a,b]\times S^1} \times M \to \mathbb{R}$, $\Phi(t,s,z) = \varphi_t(z)$.

This is a degenerate M-A equ.

Finding a geodesic φ_t $(t \in [0,1])$ between $\varphi_0, \varphi_1 \in \mathcal{K}_\omega$

$\Leftrightarrow$ solving the HCMA with Dirichlet boundary values φ_0, φ_1 along $\{0\} \times S^1 \times M$ and $\{1\} \times S^1 \times M$.

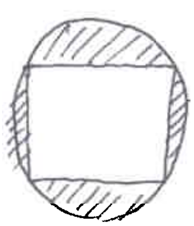
<u>Theorem</u> (X. X. Chen, 98). For any Riemann surface Σ with boundary $\partial\Sigma$ and smooth $\varphi_0 : \partial\Sigma \to \mathcal{K}_\omega$, there is a unique $C^{1,1}$-solution $\varphi : \Sigma \to \mathcal{K}_\omega$ solving HCMA with boundary value φ_0.

$\Rightarrow$ If $C_1(M) \le 0$, Chen proved lower boundedness of $\mathcal{V}_\omega$ & uniqueness by using this theorem.

<u>Q</u>: How smooth can Φ be?

<u>Examples</u>: 1) B = the unit ball in $\mathbb{C}^2$, $\varphi_0(z_1,z_2) = \left(\frac{|z_1|^2 - |z_2|^2}{2}\right)^2$

Then the unique sol. of HCMA with boundary value φ_0 is given by



$$\Phi(z_1,z_2) = \begin{cases} 0 & \text{if } |z_1|^2, |z_2|^2 \le \frac{1}{2} \\ \left(\frac{1}{2} - |z_1|^2\right)^2 & \text{if } |z_1|^2 \ge \frac{1}{2} \\ \left(\frac{1}{2} - |z_2|^2\right)^2 & \text{if } |z_2|^2 \ge \frac{1}{2} \end{cases}$$

This is only $C^{1,1}$.

2) Donaldson has an example on $D \times M$.

<u>Conclusion</u>: $C^{1,1}$ seems to be the best

Hope: partial regularity: Φ is smooth outside a small subset. If this subset is small enough, then one can still carry out the original program.

Semmes' Construction (92).

Construction of W_Ω: $\quad \rho: \tilde{U}_i \to T^*M|_{U_i}$

$(z,\lambda) \to (z, \lambda + \partial\theta_i)$, $\quad \omega|_{U_i} = \partial\bar\partial\theta_i$.

W_Ω is obtained by patching together $\tilde{U}_i$ by additive transition functions $\partial(\theta_i - \theta_j)$.

Thus ρ extends to a map: $W_\Omega \to T^*M$. let J be the pull-back of the standard complex structure on T^*M and $\Psi = \rho^*(dz \wedge d\xi)$.

Now suppose that $\varphi \in \mathcal{K}_\omega$, it induces a submanifold $\Lambda_\varphi \subset W_\Omega$: $\Lambda_\varphi|_{U_i} = \text{graph}(\partial\varphi)$.

— $Re(\Psi)|_{\Lambda_\varphi} = 0$, $\quad Im(\Psi)|_{\Lambda_\varphi} = \omega_\varphi > 0$, i.e., Λ_φ is a LS submanifold.

Basically, {exact LS submanifolds} $= \mathcal{K}_\omega$.

Prop. There is a solution Φ of HCMA on $D \times M$ iff $\exists$ a smooth family of holomorphic discs $f_p: D \to W_\Omega$ parametrized by $p \in M$ such that 1) $\pi(f_p(0)) = p \in M$, where $\pi: W_\Omega \to M$

2) $\forall z \in \partial D$, $p \in M$, $f_p(z) \in \Lambda_{\Phi(z,\cdot)}$

3) $\forall z \in D$, $p \to \pi(f_p(z))$ is a diffeo. of M.

Why: Suppose that ϕ is a solution of HCMA $\Rightarrow$ There is an integrable holo. distribution $\mathcal{D} = \ker(\omega + \partial\bar\partial\phi)$.

$\Rightarrow \forall p \in M, \exists!$ holo. leaf $=$ the graph of $h_p : D \to M$ s.t. $h_p(0) = p$.

Write $h_p(z) = \sigma_z(p)$, we get diffeo. $\sigma_z : M \circlearrowleft$ with $\sigma_0 = Id$.

— $\sigma_z^*(\omega + \partial\bar\partial\phi|_{z \times M}) = \omega$.

Define $f_p(z) = (h_p(z), \partial\phi(z, h_p(z))) \in W_\Omega$

Then $\gamma_z^* \Psi = -2\omega$, where $\gamma_z(p) = f_p(z)$.

— f_p is holo. by using HCMA.

The construction can be reversed. //

Given boundary value $\varphi_0 : \partial D \to \mathcal{X}_\omega$, we have

$$L_{\varphi_0} = \{(z, v) \in \partial D \times W_\Omega \mid v \in \Lambda_{\varphi_0(z)}\} \subset D \times W_\Omega$$

— The submanifold L_{φ_0} is totally real in $D \times W_\Omega \subset \mathbb{C} \times W_\Omega$, that is, $\xi \in TL_{\varphi_0} \Rightarrow J\xi \notin TL_{\varphi_0}$. This follows from $\Lambda_{\varphi_0(z)}$ is a LS submanifold. //

So we can consider holomorphic maps from D to $D \times W_\Omega$ which map ∂D to L_{φ_0}. This is an elliptic problem.

— Let $\bar\partial_p$ be the linearization of this elliptic problem at f_p, then $\bar\partial$ is Fredholm of index $2n$.

— If Φ is a smooth sol. and f_p is the induced holomorphic map, then M is the moduli space of holomorphic disks with the property: each f_p is super-regular, i.e., $\bar\partial_p$ has $2n$ kernel sections which span at every pt. of D.

Note that "super-regular" $\Rightarrow$ "regular", i.e. $\mathrm{Coker}(\bar\partial_\varphi) = \{0\}$.

<u>Def</u>: A solution φ of HCMA is called an almost smooth sol.
if it satisfies: 1) $\varphi \in C^{1,1}_{\text{(local)}}$
2) $\exists$ closed $E \subset D\times M$ with $\mathrm{mes}(E) = 0$ s.t. φ is smooth on $D\times M \setminus E$ and $\omega_\varphi^n \equiv 0$ along E.
3) $D\times M\setminus E$ is made of holomorphic leaves of ω_φ.

<u>Prop</u>: For every smooth $\varphi_0 : \partial D \to K_\omega$, there is a unique almost smooth solution.

<u>Basic idea</u>: $D\times M\setminus E$ consists of all super-regular holo. discs.
Or more precisely, for each boundary value $\varphi_0 : \partial D \to K_\omega$, we want to reconstruct a solution of HCMA by using the moduli of holomorphic discs and proving that this moduli has a large part of super-regular discs. Then we try to translate properties of the moduli into solutions of HCMA.
Use the continuity method.

- Bound area of holomorphic discs in terms of 1st, 2nd derivatives of boundary values. Here we need to use $C^{1,1}$-estimates of X. X. Chen.

- Prove a stronger version of Gromov's compactness theorem in this case, i.e., no bubblings.

- Introduce capacity $\mathrm{Cap}(f) = \int_D \left(\frac{\omega^n}{\omega_\varphi^n}\right)\cdot f \, dz d\bar z$.
  If $\mathrm{Cap}(f) < \infty$, then $\frac{\omega^n}{\omega_\varphi^n}$ is bounded on any compact subset $\subset D$.

  $\Rightarrow$ (i) no bubbling.
  (ii) Limit of super-regular discs with uniformly finite Capacity is super-regular

- $\int_{f \in M} \mathrm{Cap}(f) < \infty$, total integral of $\mathrm{Cap}(f)$ is finite.