1. Some $G_2$ linear algebra. Recall that $G_2$ is the subgroup of $GL(7, \mathbb{R})$ that preserves the 3-form

$$\phi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{248} - dx^{257} - dx^{356} - dx^{347},$$

where $dx^{ijk}$ means $dx^i \wedge dx^j \wedge dx^k$.

The group $G_2$ is a compact, 1-connected subgroup of $SO(7)$ that acts transitively on $S^6 \subset \mathbb{R}^7$, with isotropy group isomorphic to $SU(3)$ (in its standard representation on $\mathbb{R}^6$).

Since $\mathbb{R}^7$ is odd dimensional, a maximal torus in $G_2$ must leave a vector in $\mathbb{R}^7$ fixed and is therefore conjugate to a torus in an isotropy subgroup, i.e., $SU(3)$. Since $SU(3)$ has rank 2, it follows that $G_2$ also has rank 2 and that a maximal torus for $SU(3)$ is also a maximal torus for $G_2$. Though $SU(3)$ has a center isomorphic to $\mathbb{Z}/3\mathbb{Z}$, these central elements fix a vector in $\mathbb{R}^7$ and are therefore not central in $G_2$.

Thus, $G_2$ has trivial center, so that all of its nontrivial representations are faithful.

**Representations:** The representation ring of $G_2$ is generated by its two fundamental representations:

- The first is $V_{1,0} \simeq \mathbb{R}^7$. The second is $V_{0,1} \simeq \mathfrak{g}_2$, which has dimension 14.

The first few remaining representations are given in the following table, where the subscript is the highest weight vector and the superscript is the (real) dimension.

<table>
<thead>
<tr>
<th>$V_{0,0}$</th>
<th>$V_{1,1}$</th>
<th>$V_{1,2}$</th>
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</thead>
<tbody>
<tr>
<td>$V_{0,0}$</td>
<td>$V_{1,1}$</td>
<td>$V_{1,2}$</td>
</tr>
<tr>
<td>$V_{2,1}$</td>
<td>$V_{2,1}$</td>
<td>$V_{2,2}$</td>
</tr>
<tr>
<td>$V_{3,0}$</td>
<td>$V_{3,1}$</td>
<td>$V_{3,2}$</td>
</tr>
</tbody>
</table>

For $p \geq 0$, the representation $V_{p,0}$ is isomorphic to $S_p(\mathbb{R}^7)$, the harmonic polynomials on $\mathbb{R}^7$ of degree $p$ and $K(\mathfrak{g}_2) \simeq V_{0,2}$.

**Exterior Algebra.** The $G_2$-irreducible decompositions of the vector spaces $\Lambda^p(\mathbb{R}^7)$ will be important.
Of course $\Lambda^1(\mathbb{R}^7)$ and $\Lambda^6(\mathbb{R}^7)$ are isomorphic to $\mathbb{R}^7$ and so are irreducible. However, $\Lambda^p(\mathbb{R}^7)$ for $1 < p < 6$ are reducible.

By duality, it suffices to describe the decompositions of $\Lambda^2(\mathbb{R}^7)$ and $\Lambda^3(\mathbb{R}^7)$.

The decomposition of $\Lambda^3(\mathbb{R}^7)$ follows from the embedding of $G_2$ into $SO(7)$:

$$\Lambda^3(\mathbb{R}^7) \cong so(7) \cong g_2 \oplus g_2^1 \cong g_3 \oplus \mathbb{R}^7,$$

so we write $\Lambda^3(\mathbb{R}^7) \cong \Lambda^3_{14}(\mathbb{R}^7) \oplus \Lambda^3_{17}(\mathbb{R}^7)$.

These summands can be described explicitly as

$$\Lambda^3_{14}(\mathbb{R}^7) = \{ \ast (\alpha \wedge \ast_0 \phi_0) \mid \alpha \in \Lambda^1(\mathbb{R}^7) \}$$

$$= \{ \alpha \in \Lambda^3(\mathbb{R}^7) \mid \alpha \wedge \phi_0 = 2 \ast_0 \alpha \}$$

$$\Lambda^3_{17}(\mathbb{R}^7) = \{ \alpha \in \Lambda^3(\mathbb{R}^7) \mid \alpha \wedge \phi_0 = -\ast_0 \alpha \}$$

Similarly, there is an irreducible decomposition

$$\Lambda^3(\mathbb{R}^7) \cong \Lambda^3_{27}(\mathbb{R}^7) \oplus \Lambda^3_{23}(\mathbb{R}^7) \oplus \Lambda^3_{21}(\mathbb{R}^7),$$

where these summands have the explicit descriptions

$$\Lambda^3_{21}(\mathbb{R}^7) = \{ r \phi_0 \mid r \in \mathbb{R} \}$$

$$\Lambda^3_{23}(\mathbb{R}^7) = \{ \ast_0 (\alpha \wedge \phi_0) \mid \alpha \in \Lambda^1(\mathbb{R}^7) \}$$

$$\Lambda^3_{27}(\mathbb{R}^7) = \{ \alpha \in \Lambda^3(\mathbb{R}^7) \mid \alpha \wedge \phi_0 = \alpha \wedge \ast_0 \phi_0 = 0 \} \cong S^0_0(\mathbb{R}^7)$$

Explicitly, define $i : S^2(\mathbb{R}^7) \to \Lambda^3(\mathbb{R}^7)$ by

$$i(\alpha \wedge \beta) = \alpha \wedge \ast_0 (\beta \wedge \ast_0 \phi_0) + \beta \wedge \ast_0 (\alpha \wedge \ast_0 \phi_0).$$

Then $i(S^2_0(\mathbb{R}^7)) = \Lambda^3_{27}(\mathbb{R}^7)$. Defining $j : \Lambda^3(\mathbb{R}^7) \to S^2(\mathbb{R}^7)$ by

$$j(\gamma)(v, w) = \ast_0 ((v \wedge \phi_0) \wedge (w \wedge \phi_0) \wedge \gamma),$$

for $\gamma \in \Lambda^3(\mathbb{R}^7)$ and $v, w \in \mathbb{R}^7$, one finds that

$$j(i(h)) = -8h$$

for all $h \in S^2_0(\mathbb{R}^7)$.

2. G_2 Structures. Let $M$ be a smooth 7-manifold. Recall that a $G_2$-structure on $M$ is a 3-form $\phi$ on $M$ such that, for each point $x \in M$, there exists an isomorphism $u_x : T_x M \to \mathbb{R}^7$ such that $u_x(\phi_0) = \phi_x$.

$M$ possesses a $G_2$-structure iff $M$ is orientable and spinnable. The set of $G_2$-structures on $M$ will be denoted $\Omega^3_2(M) \subset \Omega^3(M)$. These 3-forms are the sections of an open subbundle $\Lambda^3_2(T^*M)$ of $\Lambda^3(T^*M)$.

Each $\phi \in \Omega^3_2(M)$ has an associated Riemannian metric $g_\phi$ and orientation $\ast_0 1 \in \Omega^7(M)$.

Given a $G_2$-structure $\phi \in \Omega^3_2(M)$, the $G_2$-equivariant decompositions of $\Lambda^p(\mathbb{R}^7)$ induce corresponding decompositions of $\Omega^p(M)$. For example,

$$\Omega^3_2(M, \phi) = \{ \beta \in \Omega^3(M) \mid \beta \wedge \phi = 2 \ast_0 \beta \}$$

$$\Omega^1_2(M, \phi) = \{ \beta \in \Omega^1(M) \mid \beta \wedge \phi = -\ast_0 \beta \}.$$

Recall the theorem of Fernandez and Gray:

**Theorem:** Let $\sigma$ be a $G_2$-structure on $M$. Then $\sigma$ is parallel with respect to its associated metric $g_\phi$ if and only if $d\sigma = d(\ast_0 \sigma) = 0$. 
There is a general formula for the derivatives of a $G_2$-structure:

**Proposition:** For any $G_2$-structure $\sigma \in \Omega^3_+ (M)$, there exist unique differential forms $\tau_0 \in \Omega^3(M)$, $\tau_1 \in \Omega^3(M)$, $\tau_2 \in \Omega^3(M)$, and $\tau_3 \in \Omega^3(M)$ so that the following equations hold:

\[
\begin{align*}
\text{d}\,\sigma &= \tau_0 \wedge \sigma + 3 \tau_1 \wedge \sigma + 4 \tau_2 \wedge \sigma + \tau_3 \wedge \sigma, \\
\text{d}\,\tau_0 &= \tau_1 \wedge \sigma + 2 \tau_2 \wedge \sigma + \tau_3 \wedge \sigma.
\end{align*}
\]

**Remarks:** Except for the appearance of $\tau_3$ in two places, this follows directly from the $\sigma$-decomposition of exterior forms.

For any $G \subset \text{SO}(n)$, the torsion of a $G$-structure on $M^n$ takes values in a bundle modeled on $(\text{so}(n)/g) \otimes \mathbb{R}^n$. In our case:

\[
(\text{so}(7)/g_2) \otimes \mathbb{R}^7 \simeq V_{1,0} \otimes V_{1,0} \simeq V_{0,0} \oplus V_{1,0} \oplus V_{0,1} \oplus V_{2,0}
\]

essentially by dimension count.

Recall that $K(g_2) \simeq V_{0,2} \simeq \mathbb{R}^7$, which implies Bonan’s result that a metric with holonomy in $G_2$ must be Ricci-flat.

It follows that, for the general $G_2$-structure, it must be possible to express the Ricci tensor in terms of the torsion forms $\tau_0$, $\tau_1$, $\tau_2$ and $\tau_3$. The result (got by routine calculation) is:

**Proposition** For any $G_2$-structure $\sigma \in \Omega^3(M)$, the following hold:

\[
\text{Scal}(g_\sigma) = 12 \delta \sigma \tau_1 + \frac{21}{8} \delta \sigma \tau_2 + 30 \left| \tau_1 \right|^2 - \frac{1}{2} \left| \tau_2 \right|^2 - \frac{1}{2} \left| \tau_3 \right|^2.
\]

\[
\text{Ric}(g_\sigma) = \left( \frac{1}{2} \delta \sigma \tau_3 - \frac{3}{8} \delta \sigma \tau_2 + 15 \left| \tau_1 \right|^2 - \frac{1}{4} \left| \tau_2 \right|^2 + \frac{1}{2} \left| \tau_3 \right|^2 \right) g_\sigma
\]

\[+\frac{1}{2} \left( \text{d} \, (\tau_1 \wedge * \sigma) - \frac{1}{2} \text{d} \tau_1 + \frac{1}{2} * \sigma \text{d} \tau_1 \right) - \frac{5}{8} \tau_1 \wedge * \sigma \tau_3 - \frac{1}{4} \tau_2 \wedge \tau_2
\]

\[+ \frac{1}{2} * \sigma \tau_3 + \frac{1}{4} * \sigma \tau_2 + \frac{1}{4} \sigma \tau_2 \wedge \tau_3 + \frac{1}{8} Q(\tau_3, \tau_3) \right).
\]

3. **Closed $G_2$ Structures.** From now on, I will only be considering $G_2$-structures $\sigma \in \Omega^3_+ (M)$, that are closed, i.e., $\sigma = 0$.

By the previous formulae, it follows that, for such a structure, one has $\tau_0 = \tau_1 = \tau_3 = 0$ and $\text{d}\,\tau_2 = \tau_3$.

And

\[
\tau_2 \wedge * \sigma = \tau_3 \wedge \sigma
\]

where $\tau_2$ lies in $\Omega^3_+ (M, \sigma)$. In particular, $\tau_2 \wedge * \sigma = 0$ and $\tau_3 \wedge \sigma = - * \tau_2$.

The Ricci and scalar curvature formulae simplify to

\[
\text{Scal}(g_{\sigma}) = -\frac{1}{2} \left| \tau_2 \right|^2
\]

\[
\text{Ric}(g_{\sigma}) = \frac{1}{4} \left| \tau_2 \right|^2 g_{\sigma} - \frac{1}{4} \left( \text{d} \tau_2 - \frac{1}{2} * \sigma \tau_2 \wedge \tau_2 \right).
\]

In particular, note that the scalar curvature is pointwise non-positive and vanishes identically if and only if $\sigma$ is also coclosed.
These formulae show that $g_\alpha$ is Einstein if and only if
\[ d\tau_\alpha = \frac{3}{11} |r|_2^2 \sigma + \frac{1}{2} \ast_\sigma (\tau_2 \wedge \tau_2) \]

Using this, Cleret and Ivanov (math.DG/0306362) have recently shown that any closed $G_2$-structure on a compact manifold whose associated metric is Einstein must actually be co-closed as well.

**Hitchin's Volume Functional and Flow.** Suppose that $\mathcal{M}$ is compact and let $S \in H^3_{dR}(M, \mathbb{R})$ be a cohomology class. Define
\[ Z_+(S) = \{ \sigma \in \Omega^3_+(M) \mid d\sigma = 0, [\sigma] = S \} \]
as the set of closed $G_2$-structures whose de Rham cohomology class is $S$. Note that $Z_+(S)$ is an open subset of $S$ (which is an affine subspace of $Z^3(M)$, the space of closed 3-forms on $M$).

Hitchin defined a function $f : Z_+(S) \to \mathbb{R}^+$ by
\[ f(\sigma) = \int_M \ast_\sigma 1 = \int_M \sigma \wedge \ast_\sigma \sigma. \]

**Proposition:** (Hitchin) $\sigma \in Z_+(S)$ is a critical point of $f$ if and only if $d \ast_\sigma \sigma = 0$. All the critical points of $f$ are nondegenerate, modulo the action of the diffeomorphism group. The gradient flow of the functional $f$ is given by
\[ \frac{d}{dt}(\sigma) = \Delta_\sigma \sigma = d(\delta_\sigma \sigma) \]

Steve Altschuler and I had considered this (transversely) parabolic flow in 1992 with an eye towards trying to construct compact manifolds with holonomy $G_2$. Here are some of the results that we derived about it.

From now on, write $\tau$ instead of $\tau_2$, for simplicity. We have
\[ d\sigma = 0, \quad \text{and} \quad d\ast_\sigma \sigma = \tau \wedge \sigma \]
where $\ast_\sigma (\tau \wedge \sigma) = -\tau$. One can now easily compute that
\[ d\tau = \frac{1}{3} |\tau|^2 \sigma + \gamma \]
for some $\gamma \in \Omega^3_+(M, \sigma)$.

The evolution equation (Hitchin’s flow) becomes
\[ \frac{d}{dt}(\sigma) = d\tau. \]

The formulae from the previous page then imply
\[ \frac{d}{dt}(\ast_\sigma 1) = \frac{1}{3} |\tau|^2 \ast_\sigma 1. \]

Note that the volume form is increasing pointwise, and not just on average (as would be expected for the $f$-gradient flow). Thus,
\[ \frac{d}{dt}(f(\sigma(t))) = \frac{1}{3} \int_M |\tau|^2 \ast_\sigma 1 \]

Computing a further derivative and integrating by parts yields the formula
\[ \frac{d^2}{dt^2}(f(\sigma(t))) = \frac{1}{3} \frac{d}{dt} \int_M |\tau|^2 \ast_\sigma 1 = \int_M \left( \frac{2}{3} |\tau|^2 - \frac{3}{2} |\tau|^2 \right) \ast_\sigma 1. \]