MSRI — von Neumann workshop
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Crepant Resolutions of CY orbifolds

August/15th/200

Calabi-Yau : X Kähler mfld w/ $K_X = 0$. $\quad K_X = \Lambda^{0,n}(TX)$

Ω holomorphic volume form.
nowhere vanishing

singular Calabi-Yau

- orbifold singularity : the easiest kind of singularity

locally $\mathbb{C}^n/G$ $\quad$ $\underline{G \leq SL(n;\mathbb{C})}$ finite subgroup.
(in order to have CY str.).

$$X \xrightarrow{\pi} \mathbb{C}^n/G \quad \text{resolution of singularities}$$

$$K_X = K_{\mathbb{C}^n/G} + \text{discrepancy}$$

X is Calabi-Yau $\iff$ $K_X = 0$.

i.e.

no discrepancy = crepant.

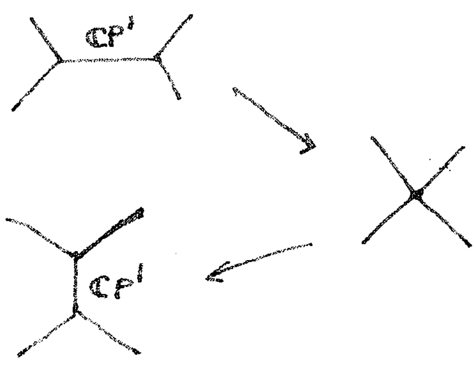
<u>Question 1</u> : When does a crepant resolution exist? Is it unique?

<u>Question 2</u> : How does the finite group G describe the topology of X?

<u>Answer 1</u> : $n=2$ Always exists and it is unique.

$n=3$ Exists but not unique.

<u>Reason</u> : flops



<u>$n=4$</u>. Might not exist

<u>Reason</u> : terminal sing^s.

$\boxed{n=2}$ $G \leq SL(2;\mathbb{C})$ finite group.

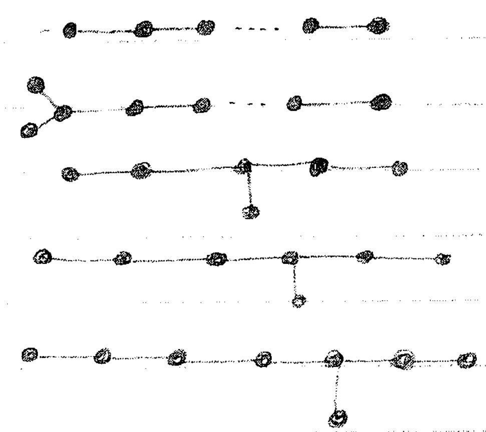
$\mathbb{C}^2/G$ classified by Klein 1884

Kleinian singularities / rational double pts / du Val sing^s

5 families of such G : $\xleftrightarrow{\text{McKay Corresp}}$ simply laced Dynkin diagrams.

2 inf. families:
- C_k the cyclic subgroup
- D_k the binary dihedral gr. order $=4k$

3 exceptional:
- T the binary ~~tetrahedral~~ order $=24$
- O the binary octahedral order $=48$
- I the binary icosahedral order $=120$



G $R_0, R_1, \dots, R_r$ irreducible representations

$$\mathbb{C}^2 \otimes R_i = \sum a_{ij} R_j \qquad a_{ij} = 0 \text{ or } 1$$

$\Rightarrow$ the extended Dynkin diag.

Moreover : $A = [a_{ij}]$ the adjacency matrix

$A = 2I - C$ $\quad C =$ the Cartan matrix associated to the Dynkin diagram.

This correspondence completely describes the topology of the crepant resolution $X \longrightarrow \mathbb{C}^2/G$.

$$\pi^{-1}(0) = \bigcup_{i=1}^{r} C_i$$

C_i rational curves intersecting transversally.

$C_i^2 = -2$.

the dual of the Dynkin diagram.

Moreover : $\{C_i\}$ basis $H_2(X)$

$$[C_i \cdot C_j]_{i,j=\overline{1,r}} = C.$$

Cohomology:

R_i irreducible representations $\longmapsto$ $\mathcal{R}_i$ holomorphic vector bundles over X.

$\{c_1(\mathcal{R}_i)\}$ basis $H^2(X)$ dual to $\{C_i\}$.

$$\int_{C_j} c_1(\mathcal{R}_i) = \delta_{ij}.$$

product: $\left[\int_X c_1(\mathcal{R}_i)\cdot c_1(\mathcal{R}_j)\right] = C^{-1}$ $\longleftarrow$ Geometrical Interpreta of the McKay Correspond

Gonzalez-Sprinberg & Verdier : case-by-case analysis

Kronheimer & Nakajima : gauge theory techniques.

$\boxed{n=3}$ A crepant resolution exists but it is not unique

$$X \longrightarrow \mathbb{C}^3/G.$$

the same: Euler number $\longleftarrow$ stringy orbifold Euler & Betti numbe

Betti numbers

- DHVW
- entirely described in terms of (

ex: $e(X) = \#$ of conj. classes of G

Existence : early 90's: Ito-Roan-Markushevich case-by-case analys

'95 : Nakamura $\mathrm{Hilb}^G(\mathbb{C}^3)$ is a crepant resolution

- Ito & Nakamura : proved for abelian gr^s.
- Bridgeland, King & Reid : the general case

derived categories techniques.

'02 : Craw & Ishii all the crepant resolutions of $\mathbb{C}^3/G$ have a moduli space description (proved for G abelian)

How to differentiate between different crepant resolutions?

$X \longrightarrow \mathbb{C}^3/G$ crepant resolution ($\mathrm{Hilb}^G(\mathbb{C}^3)$ or Craw & Ishii's situation)

R_i irred. repr. of G $\rightsquigarrow$ $\mathcal{R}_i$ holomorphic vector bdl. $/X$.

$R_i \otimes \mathbb{C}^3 = \sum a_{ij} R_j$
$R_i \otimes \Lambda^2\mathbb{C}^3 = \sum b_{ij} R_j$ $\quad C = [a_{ij} - b_{ij}]$ generalized Cartan matrix.

BKR's techniques $\Rightarrow$ $\mathcal{R}_i$ form a basis of $K(X)$

$ch(\mathcal{R}_i)$ basis of $H^*(X)$.

Our idea: Use index theory to obtain information about multiplicative structure.

Need to understand the geometry of X.

Assumption: G acts w/ an isolated singularity on $\mathbb{C}^3$.

Thm 1: X has a Ricci-flat ALE metric.

Idea of the proof: based on work of Sardo-Infirri (generalizing Kronheimer's construction of ALE gravitational instantons)

Step 1: realize X as a symplectic reduction

Step 2: use the proof of the Calabi conjecture for ALE mflds (Tian-Yau / Joyce).

$\mathcal{E}$
$\downarrow$
X holomaphic vector bundle w/ E = fiber at infinity

Dirac operator:

$$D^+ : C^\infty(X; S^+ \otimes \mathcal{E}) \longrightarrow C^\infty(X; S^- \otimes \mathcal{E})$$

APS: $$\text{index } D^+ = \underbrace{\int_X ch(\mathcal{E})\, \hat{A}(X)} - \frac{\eta_E}{2}$$

$\underline{\mathcal{E} = \mathcal{R}_i \otimes \mathcal{R}_j^*}$ $\longrightarrow$ info. about multiplicative str.

Thm 2: ~~[illegible]~~ ~~[illegible]~~ In a very special completion

$$\text{index } D^+ = 0.$$

~~Prop~~:
$$\eta_E = \frac{1}{|G|} \sum_{\substack{g \in G \\ g \neq id}} \frac{\chi_E(g)}{-\chi_{\mathbb{C}^3}(g) + \chi_{\Lambda^2 \mathbb{C}^3}(g)}$$

spectral invariant

Consequence: $$\left[\int_X \left(ch(\mathcal{R}_i) - rk(\mathcal{R}_i)\right)\left(ch(\mathcal{R}_j) - rk(\mathcal{R}_j)\right) \right]$$

$$= C^{-1}$$

Notice: 1. this does not depend on the crepant resolution we choose.

2. the formula does not give info about $\int_X c_1^3(\mathcal{R}_i)$.

(Yukawa coupling)

Work in progress :

$$\int_X c_1^3(\mathcal{R}_i) + \underbrace{\int_X c_1(\mathcal{R}_i)\, c_2(X)} = \frac{\ell_{R_i}}{2}.$$

$$\int_X c_1(\mathcal{R}_i)\, c_2(X) = \sum_{\substack{D_g \\ \text{crepant} \\ \text{divisors}}} \underbrace{\alpha(R_i, g)}_{\substack{\uparrow \\ \text{independent} \\ \text{of } X}} \; \underbrace{c_2(D_g)}_{\substack{\uparrow \\ \text{depends} \\ \text{on } X}}.$$

where

$$R(G) \times G \xrightarrow{\alpha} \mathbb{Z}$$
$$(R, g) \longmapsto \alpha(R, g)$$

$$R(g) = \varepsilon^{\alpha(R,g)}.$$