1 Representations of Surface Groups

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A *surface group* is the fundamental group of a compact surface

$$\Gamma = \pi_1(S).$$

We study homomorphisms (*representations*)

$$\rho : \Gamma \to PSL(n, \mathbb{R});$$

and more precisely representations in a preferred connected component of the space of representations

$$\mathit{Rep}^*(\Gamma, PSL(n, \mathbb{R})) = \{\rho : \Gamma \to PSL(n, \mathbb{R})\}/PSL(n, \mathbb{R}).$$

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2 Examples and definitions

1) Fuchsian representations in $PSL(2, \mathbb{R})$. Riemann uniformisation theorem:

$$\begin{array}{c} \text{complex structure } J \text{ on } S \rightsquigarrow \rho : \Gamma \to PSL(2, \mathbb{R}) \\ \text{s.t. } (S, J) \text{ is conformal to } \mathbb{H}^2/\rho(\Gamma) \end{array} .$$

Such a representation is said to be *Fuchsian*, it is the *monodromy of a hyperbolic structure.*

2) Fuchsian representations in $PSL(n, \mathbb{R})$. By definition these are compositions

$$\Gamma \overset{\text{Fuchsian}}{\longrightarrow} PSL(2, \mathbb{R}) \overset{\text{irreducible}}{\longrightarrow} PSL(n, \mathbb{R}).$$

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2) Hitchin representations in $PSL(n, \mathbb{R})$: a representation which can be deformed to a Fuchsian representation.

3) Hitchin component $\mathcal{H}(n)$ is (one of) the connected component(s) of

$$\{\text{Hitchin representations}\}/PSL(n, \mathbb{R}).$$

Theorem [Hitchin] *The Hitchin component is homeomorphic to a ball of dimension $-\chi(S)(n^2 - 1)$.*

Question: Do Hitchin representations have nice properties (faithful, discrete) ? Are they "monodromies of geometric structure" in some sense ?

Answer: Yes, and furthermore there seem to be a "Higher Teichmüller" theory interpreting these components in other mathematical dialects.

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3 A Conjecture about Uniformisation

- Let's fix a complex structure J on S, Hitchin constructed a homeomorphism
$$\phi_J : H^0(K_J^2) \oplus \ldots \oplus H^0(K_J^n) \to \mathcal{H}(n)_{= \text{ Hitchin component}}.$$

- We want to get rid of the choice of J. Let $\mathcal{T}(S)$ be Teichmüller space. Let $E \to \mathcal{T}(S)$ be the vector bundle with fibre
$$E_J = H^0(K_J^3) \oplus \ldots \oplus H^0(K_J^n).$$
Let
$$\Phi : \begin{cases} E & \mapsto & \mathcal{H}(n) \\ (J,\omega) & \to & \phi_J(\omega) \end{cases}$$
Notice that Φ is equivariant under the action of the Mapping Class Group $\mathcal{M}(S) = Out(\Gamma)$.

Conjecture *The map Φ is a homeomorphism*

- It is true for $n = 2$ (trivial), and for $n = 3$ (using results about affine spheres).
- Φ is surjective.
- This conjecture has a translation as the stability of a minimal surface in a symmetric space.
- This would imply $\mathcal{H}(n)/\mathcal{M}(S)$ is naturally a complex manifold.

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4 Corollary of the Main Result

Theorem *Every Hitchin representation is discrete, faithful, and "purely loxodromic". Finally the Mapping Class Group $\mathcal{M}(S)$ acts properly on $\mathcal{H}(n)$*

- ρ purely loxodromic =
$$\forall \gamma \in \Gamma, \gamma \neq id, \rho(\gamma)_{\text{has only real eigenvalues with multiplicity 1}}.$$

- A priori inaccessible by Hitchin's techniques.
- The above "uniformisation" conjecture describes the topology on $\mathcal{H}(n)/\mathcal{M}(S)$.
- What about geometric structures ?

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5 What happens in low dimensions ?

1) For n=2 : The Hitchin component is Teichmüller space, which corresponds to hyperbolic structures on S.

2) For n=3 :

Theorem [Choi.S.Y-W.Goldman] *Every Hitchin representation in $PSL(3,\mathbb{R})$ preserves a convex open set Ω in $\mathbb{P}(\mathbb{R}^3)$, such that $\Omega/\rho(\Gamma)$ is homeomorphic to S. A Hitchin representation is the monodromy of a convex real projective structure on S.*

- So far, this is over, no available 2-dimensional homeogeneous space for $SL(4,\mathbb{R})$. No classical geometric structure underlies $\mathcal{H}(4)$.
- But, there is a notion of *"dynamical geometric structures"* on US (with coordinate charts) whose monodromies are Hitchin representations. But, they are long to define, and I prefer to describe some geometric consequences, and relate Hitchin representations to
  (1) *curves in the projective space*
  (2) *generalised crossratios on S^1*

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6 The boundary at infinity of Γ

The boundary at infinity $\partial_\infty\Gamma$ of Γ is a topological circle on which Γ acts. It is is defined as the "horizon" of the group Γ viewed as a geometric object. It has some natural geometric realisations.

- *Hyperbolic structure* If S is equipped with a hyperbolic structure. $\partial_\infty\Gamma$ is identified with $\partial_\infty\mathbb{H}^2$, the boundary at infinity of the hyperbolic plane,
- For a Hitchin representation ρ in $PSL(2,\mathbb{R})$, we have a ρ-equivariant injective map from $\partial_\infty\Gamma$ to $\mathbb{P}(\mathbb{R}^2) = \partial_\infty\mathbb{H}^2$
- *Convex real projective structure* If $\Omega \subset \mathbb{P}(\mathbb{R}^3)$ is such that $\Omega/\rho(\Gamma) = S$. Then $\partial_\infty\Gamma$ is identified with the convex curve $\partial\Omega$
- For a Hitchin representation ρ in $PSL(3,\mathbb{R})$, we have a ρ-equivariant convex map from $\partial_\infty\Gamma$ to $\mathbb{P}(\mathbb{R}^3)$

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7 Hyperconvex curves

A continuous curve ξ from S^1 to $\mathbb{P}(\mathbb{R}^n)$ is *hyperconvex* if for any distinct points $(x_1,\ldots,x_n)$ in S^1 the following sum is direct

$$\xi(x_1) \oplus \ldots \oplus \xi(x_n).$$

- for $n = 2$ hyperconvex means injective,

- for $n = 3$ hyperconvex means convex,
- The Veronese embedding is hyperconvex.
- By Cauchy-Crofton formula, hyperconvex curves are rectifiable with universally bounded length :

$$\text{length(c)} = \int_{\text{hyperplanes P}} \underbrace{\sharp(c \cap P)}_{\leq n-1} dP.$$

Theorem *If ρ is a Hitchin representation, then there exists a (unique) ρ-equivariant hyperconvex curve ξ, the* limit curve*, from $\partial_\infty \Gamma$ to $\mathbb{P}(\mathbb{R}^n)$. Furthermore, these (generally only C^1) curves admits continuous osculating flags in general position*

- for Fuchsian representations, we get the Veronese embedding,
- Hitchin representations are symmetries of hyperconvex curves.
- By definition *osculating flag* $\eta = (\eta_1, \ldots, \eta_{n-1})$ is such that

$$\eta_p(x) = \lim_{(x_1,\ldots,x_p)\to x} \xi(x_1) \oplus \ldots \oplus \xi(x_p).$$

- *Universal Hitchin component* (when genus goes to infinity)=$\{$hyperconvex curves$\}$

Idea of the proof 1) *Openness* After a small deformation the limit curve persists to exist. This is a consequence of the stability of hyperbolic dynamical systems. It is a general fact. Compare with the case of quasi-Fuchsian representations in $SL(2, \mathbb{C})$.

Difficult part : to prove the (a priori C^0) perturbed curve is still hyperconvex.

2) *Closeness* After a large perturbation, the limit curve continues to exist. Here use the rectifiability of the curve. Again, compare with the quasi-Fuchsian case in $SL(2, \mathbb{C})$.

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8 Crossratios and Periods

A *crossratio* on a set $\partial_\infty \Gamma$ is a Hölder function $b : S^4 \setminus \Delta \to \mathbb{R}^*$, invariant under the action of Γ satisfying

$$\begin{aligned} b(x, y, z, t) &= b(x, y, z, w)b(x, w, z, t) \\ b(x, y, z, t) &= b(x, y, w, t)b(w, y, z, t) \\ b(x, y, z, t) &= b(x, t, z, y)^{-1} \\ b(x, y, z, t) &= b(z, t, x, y) \end{aligned}$$

Let $\gamma \in \Gamma$. The *period* of γ is

$$l_b(\gamma) = \log |b(\gamma^+, y, \gamma^-, \gamma y)|.$$

Where γ^+ (resp. γ^-) the attracting (resp. repelling) fixed point of γ on $\partial_\infty \pi_1(S)$, y any element.

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9 Examples

- The *classical* crossratio on $\mathbb{P}(\mathbb{R}^2)$,

$$\rho \in \mathcal{H}(2) \rightsquigarrow \text{ crossratio on } \partial_\infty\Gamma \simeq \mathbb{P}(\mathbb{R}^2).$$

satisfying an extra rational functional relation F(2).

- if ξ and ξ^* are curves from S^1 to $\mathbb{P}(E)$ and $\mathbb{P}(E^*)$ respectively,

$$b(x,y,z,t) = \frac{\langle\hat{\xi}(x), \hat{\xi}^*(y)\rangle\langle\langle\hat{\xi}(z), \hat{\xi}^*(t)\rangle}{\langle\hat{\xi}(z), \hat{\xi}^*(y)\rangle\langle\hat{\xi}(x), \hat{\xi}^*(t)\rangle}.$$

(Generalisation to flag manifolds)

- *Geodesic flows* of negatively curved metric on $S \rightsquigarrow$ crossratios Periods = length of closed geodesics. (In general, Anosov flows $\leftrightsquigarrow$ Crossratios)

Theorem *There is a *bijection* between $\mathcal{H}(n)$ and the set of crossratios satisfying functional relations $F(n)$. Under this correspondance*

$$l_b(\gamma) = \log(\frac{\lambda_{max}(\rho(\gamma))}{\lambda_{min}(\rho(\gamma))}).$$

Let n tends to infinity

$$\mathcal{H}(n) \hookrightarrow \{\text{crossratio on } \partial_\infty\Gamma\}\text{"} \hookrightarrow \text{"}Rep(\Gamma, Symp(T^*S^1)) \stackrel{?}{=} \mathcal{H}(\infty)$$

F(2)

$$b(e,f,g,h) = \frac{(b(e,u,v,w) - b(f,u,v,w))(b(g,u,v,w) - b(h,u,v,w))}{(b(e,u,v,w) - b(h,u,v,w))(b(g,u,v,w) - b(f,u,v,w))}.$$

F(n) involves quotient of determinants of matrices whose coefficients are crossratios

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10 Dynamical geometric structures

Before proceeding to the definition, we recall the definition of a *contracting (or dilating) bundle* over a dynamical system.

Let X be a topological space equipped with a flow ϕ_t. Let E be a topological vector bundle over X such that the action of ϕ_t lifts to an action of a flow ψ_t by bundle automorphisms. Assume E is equipped a metric g. The bundle E is *contracting* (resp. *dilating*), if there exist positive constants A and B, such that for every u in E, for every t such that $t > 0$ (resp. $t < 0$)

$$\|\psi_t(u)\| \leq Ae^{-B|t|}.$$

It is useful and classical to remark that if X is compact,

1. the metric g plays no role;
2. the parametrisation of the flow plays no role either, that is if we change the parametrisation of the flow the bundle will stay to be contracting for this new flow.

Therefore to be contracting or dilating over a compact topological space X is a property of the orbit lamination $\mathcal{L}$, the bundle E and the "parallel transport" on E along leaves of $\mathcal{L}$.

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11 (M, G)-Anosov structure

Let M be a manifold equipped with a pair of continuous foliations $\mathcal{E}^{\pm}$, whose tangential distributions are $E^{\pm}$, and such that

$$TM = E^{+} \oplus E^{-}.$$

Let G be a Lie group of diffeomorphisms preserving these foliations.

Let V be a manifold equipped with an Anosov flow ψ_t. Let $\mathcal{L}$ be the orbit foliation. Let $\tilde{V}$ be a Galois covering with covering group Γ.

We shall say V is *A-modelled* on M, if there exists a representation ρ of Γ in G, the *holonomy representation*, a continuous map F from $\tilde{V}$ to M, the *developing map* enjoying the following properties

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- Γ-*equivariance*:
$$\forall \gamma \in \Gamma,\ F \circ \gamma = \rho(\gamma) \circ F,$$
- *Flow invariance*:
$$F \circ \psi_t(x) = F(x),$$
- *Hyperbolicity*: We consider the induced bundle $F^{\pm} = F^{*}E^{\pm}$. By the flow invariance, these bundles are equipped with a parallel transport along the orbit of ψ_t, and by Γ-equivariance this parallel transport is invariant under Γ. Our last hypothesis is that the corresponding lift of the action of ψ_t on F^{+} (resp. on F^{-}) is contracting (resp. dilating).

We also say $(V, \mathcal{L})$ admits a (M, G)-*Anosov structure*.

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12 The case of Hitchin representations

- $G = PSL(n, \mathbb{R})$
- $M = PSL(n, \mathbb{R})/\{\text{diagonal matrices}\}$, a point in M is a family of n lines $\mathbb{L} = \{L_i\}_{i \in \{1,\ldots,n\}}$ in direct sum.
- Foliations come from the product structure
$$M \subset Flag \times Flag.$$
- $V = US \simeq \partial_\infty \Gamma^3 \setminus \Delta$ with the geodesic flow lamination, defined only using $\partial_\infty \Gamma$.