Flat Lagrangian submanifolds in $\mathbb{C}^n$ and $\mathbb{C}P^n$

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Acknowledgments

I would like to thank
Chuu-Lian Terng, my advisor and mentor
IPAM in UCLA for financial support (Spring 2003)

§1. Background

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"Flat" means the Riemann curvature tensor of the induced metric on M is 0. Since Riemann proved that a flat Riemannian manifold is locally isometric to the Euclidean space, these flat submanifolds can also be viewed as Lagrangian isometric immersion of a domain in the Euclidean space.

Moore and Morvan applied Cartan-Kähler theory in [MM01] to prove that the system of equations for local isometric Lagrangian immersions of a Riemannian manifold in $\mathbb{C}^n$ is over-determined when $n \geq 3$ and most Riemannian manifolds could not admit such immersions even locally. However they show that there is a plentiful supply of flat Lagrangian submanifolds:

Theorem [MM01]: Let p be a point in $\mathbb{E}^n$. The isometric Lagrangian immersions from an open neighborhood U of p into $\mathbb{C}^n$ depend upon $n(n+1)/2$ functions of a single variable.

On the other hand, inspired by Tenenblat and Terng's generalization of the classical Bäcklund transformation ([TT80, Ten85]), Dajczer and Tojeiro generalized sphere congruence and Ribaucour transformation to higher dimensions. They have successfully constructed Ribaucour transformations for flat n-submanifolds of S^{2n-1} in [DT95], and later for flat Lagrangian submanifolds of $\mathbb{C}^n$ and $\mathbb{C}P^{n-1}$ in [DT00].

We'll identify these transformations as dressing actions, as Terng and Uhlenbeck did for classical Bäcklund transformations in [TU00].

§2. The n-dim'l system associated to symmetric space

Let τ be a conjugate linear involution of any complex semi-simple Lie algebra $\mathcal{G}$, σ a complex linear involution of $\mathcal{G}$ such that $\tau\sigma = \sigma\tau$, $\mathcal{U}$ the fixed point set of τ, and $\mathcal{U}_0$ the subalgebra of $\mathcal{U}$ fixed by σ. Let $\mathcal{U} = \mathcal{U}_0 + \mathcal{U}_1$ denote the Cartan decomposition of the symmetric space U/U_0. Let $\mathcal{A}$ be a maximal abelian linear subspace of $\mathcal{U}_1$, and $a_1, \cdots, a_n$ a basis of $\mathcal{A}$. Let $\mathcal{A}^\perp$ denote the orthogonal complement of $\mathcal{A}$ with respect to the Killing form of $\mathcal{U}$.

The U/U_0-system is the following system for $v : \mathbb{R}^n \to \mathcal{A}^\perp \cap \mathcal{U}_1$:

$$[a_i, v_{x_j}] - [a_j, v_{x_i}] = [[a_i, v], [a_j, v]], \quad i \neq j.$$

Or equivalently, the following connection 1-form is flat for all $\lambda \in \mathbb{C}$:

$$\theta_\lambda = \sum_{i=1}^{n} (a_i \lambda + [a_i, v])\, \mathrm{d}x_i,$$

which satisfies the U/U_0-*reality condition*:

$$\tau(\theta_\lambda) = \theta_{\bar{\lambda}}, \qquad \sigma(\theta_\lambda) = \theta_{-\lambda}.$$

We call θ_λ a *Lax pair* if $n = 2$, and a *Lax n-tuple* for general n.
As proved in [Te02], the U/U_0-system is independent of the choice of basis of $\mathcal{A}$, and is essentially given by the first commuting n-flows in the U/U_0-hierarchy.

Let U/U_0 be the symmetric space $\mathrm{U}(n)/\mathrm{O}(n)$. Then $\mathcal{U} = \mathrm{u}(n)$, $\mathcal{U}_0 = \mathrm{o}(n)$, and

$$\mathcal{U}_1 = \{-\mathrm{i}F \mid F = (f_{ij}) \in \mathrm{gl}(n, \mathbb{R}), f_{ij} = f_{ji}\}.$$

The linear subspace $\mathcal{A}$ spanned by

$$\{a_j = \mathrm{i}e_{jj} \mid 1 \leq j \leq n\}$$

is a maximal abelian subspace in $\mathcal{U}_1$, and

$$\mathcal{U}_1 \cap \mathcal{A}^{\perp} = \{-\mathrm{i}F \mid F = (f_{ij}) \in \mathrm{gl}(n, \mathbb{R}),\ f_{ij} = f_{ji},\ f_{ii} = 0\}.$$

The corresponding U/U_0-system written in terms of F $(v = -\mathrm{i}F)$, i.e., the $\mathrm{U}(n)/\mathrm{O}(n)$-system is

$$\begin{cases} (f_{ij})_{x_i} + (f_{ij})_{x_j} + \sum_k f_{ik} f_{jk} = 0, \text{if} \quad i \neq j, \\ (f_{ij})_{x_k} = f_{ik} f_{kj}, \text{if } i, j, k \text{ are distinct.} \end{cases}$$

Or equivalently, *Lax n-tuple* $\mathrm{i}\lambda\delta + [\delta, F]$ is flat for all $\lambda \in \mathbb{C}$, where $\delta = \mathrm{diag}(\mathrm{d}x_1, \cdots, \mathrm{d}x_n)$.
It is important to note that the above system implies $(\sum_k \frac{\partial}{\partial x_k}) f_{ij} = 0$.

§3. Geometry of the $\mathrm{U}(n)/\mathrm{O}(n)$-system

Let $\langle\ ,\ \rangle$ and w be the standard inner product and symplectic form on $\mathbb{C}^n = \mathbb{R}^{2n}$ respectively, i.e.,

$$\langle X, Y\rangle = \mathrm{Re}(\bar{X}^t Y), \quad w(X,Y) = \mathrm{Im}(\bar{X}^t Y), \ \ X, Y \in \mathbb{C}^n.$$

$A = B + \mathrm{i}C \in \mathrm{gl}(n, \mathbb{C})$ can be identified as $\left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right)$ in $\mathrm{gl}(2n, \mathbb{R})$. This identifies $\mathrm{u}(n)$ as the following subalgebra of $\mathrm{o}(2n)$:

$$\mathrm{u}(n) = \left\{ \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathrm{o}(2n) \;\middle|\; B \in \mathrm{o}(n), C \in \mathrm{gl}(n, \mathbb{R}) \text{ symmetric} \right\}.$$

The standard complex structure on $\mathbb{R}^{2n}$ is $J = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$. The group $\mathrm{U}(n)$ can be identified as the elements of $\mathrm{O}(2n)$ that commute with J, i.e., also take the above special form.

Lemma: Let $X : U \to \mathbb{R}^{2n}$ be a Lagrangian submanifold, and $(e_1, \cdots, e_n)$ a local orthonormal tangent frame. Then $(Je_1, \cdots, Je_n)$ is an orthonormal normal frame. Moreover, let $g = (e_1, \cdots, e_n, Je_1, \cdots, Je_n)$. Then $g^{-1}\,\mathrm{d}g$ is a $\mathrm{u}(n)$-valued 1-form, i.e., it is of the form $\begin{pmatrix} \xi & -\eta \\ \eta & \xi \end{pmatrix}$ where ξ is an $\mathrm{o}(n)$-valued 1-form and η is 1-form with value in the space of symmetric matrices. Conversely, if M^n has a local orthonormal frame $g = (e_1, \cdots, e_n, e_{n+1}, \cdots, e_{2n})$ such that $e_1, \cdots, e_n$ are tangent to M and $g^{-1}\,\mathrm{d}g$ is $\mathrm{u}(n)$-valued 1-form, then M is Lagrangian for some constant complex structure J' of $\mathbb{R}^{2n}$.

Definition: An *Egorov metric* is a flat metric on a domain $\mathcal{O} \subset \mathbb{R}^n$ taking the following form

$$\mathrm{d}s^2 = \sum_{i=1}^{n} \phi_{x_i} \, \mathrm{d}x_i^2$$

for some smooth function $\phi : \mathcal{O} \to \mathbb{R}$. If in addition $\sum_i \phi_{x_i} = 1$, we will call it *spherical Egorov metric*. The function ϕ will be called the *potential* of the metric.

These special classes of orthogonal coordinate systems were extensively studied by Darboux, Bianchi and Egorov. Also see [TU98] for relations with Frobenius manifolds and WDVV equation.

Lemma: Given a solution F of the $\frac{\mathrm{U}(n)}{\mathrm{O}(n)}$-system, and $b_{10}, \cdots$, b_{n0} smooth positive functions of one variable, one can solve

$$\begin{cases} (b_i)_{x_j} = f_{ij} b_j & i \neq j, \\ b_i(0, \cdots, 0, x_i, 0, \cdots, 0) = b_{i0}(x_i). \end{cases}$$

Then $\sum_{i=1}^n b_i^2 \, \mathrm{d}x_i^2$ will be an Egorov metric. Its potential ϕ can be solved from the system $\phi_{x_i} = b_i^2$.

Conversely, given the potential ϕ of an Egorov metric, define $f_{ij} = \frac{\phi_{x_i x_j}}{2\sqrt{\phi_{x_i}\phi_{x_j}}}$ if $i \neq j$, and $f_{ii} = 0$. Then the Levi-Civita connection 1-form for the metric is given by $w_{ij} = -f_{ij}(\,\mathrm{d}x_i - \,\mathrm{d}x_j)$, and the flatness of the metric gives exactly the $\frac{\mathrm{U}(n)}{\mathrm{O}(n)}$-system for $F = (f_{ij})$.

Fundamental Theorem [DT00, Te02]: Let M^n be a flat Lagrangian submanifold of $\mathbb{C}^n$ with non-degenerate normal bundle. Then there exist global line of curvature coordinates $x_1, \cdots, x_n$, parallel normal frame $e_{n+1}, \cdots, e_{2n}$, an $O(n)$-valued map $A = (a_{ij})$, and a map $b = (b_1, \cdots, b_n)$ such that the fundamental forms of M are $\mathrm{I} = \sum_{i=1}^n b_i^2 \, \mathrm{d}x_i^2$ (Egorov metric), $\mathrm{I\!I} = \sum_{i,j=1}^n b_i a_{ji} \, \mathrm{d}x_i^2 e_{n+j}$. Moreover, let $f_{ij} = (b_i)_{x_j}/b_j$, $f_{ii} = 0$. Then $F = (f_{ij})$ is a solution of the $\frac{\mathrm{U}(n)}{\mathrm{O}(n)}$-system. Conversely, given $(F, b_{10}, \cdots, b_{n0})$ as in previous lemma, there exists a (unique up to $\mathrm{U}(n) \ltimes \mathbb{C}^n$) flat Lagrangian submanifold of $\mathbb{R}^{2n}$ with non-degenerate normal bundle so that the corresponding solution of the $\frac{\mathrm{U}(n)}{\mathrm{O}(n)}$-system is F and the first fundamental form is the Egorov metric $\mathrm{I} = \sum_{i=1}^n b_i^2 \, \mathrm{d}x_i^2$ in the lemma. Furthermore, these submanifolds lie in S^{2n-1} if and only if $\sum_i \phi_{x_i} = 1$, i.e., the first fundamental form is a spherical Egorov metric.

Sketch of proof: Let $W = \left(\begin{smallmatrix} E & X \\ 0 & 1 \end{smallmatrix}\right)$, and

$$\theta_\lambda = \begin{pmatrix} \mathrm{i}\lambda\delta + [\delta, F] & \delta b \\ 0 & 0 \end{pmatrix} \qquad \text{where } b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

Now one can solve W from $W^{-1}\,\mathrm{d}W = \theta_\lambda$ uniquely up to rigid motions. Then it is directly verified that X will give a flat Lagrangian submanifold in $\mathbb{C}^n$.

We have obtained the following:

Space of Egorov metrics

$\cong$ Space of $(F, b_{10}, \cdots, b_{n0})$

$\cong$ Space of local flat Lagrangian submanifolds in $\mathbb{C}^n$ with non-degenerate normal bundle modulo $\mathrm{U}(n) \ltimes \mathbb{C}^n$

Élie Cartan proved that a flat n-dimensional submanifold can not be locally isometrically immersed in S^{n+k} if $k < n-1$, but can be locally isometrically immersed into S^{2n-1}.

Fact: Let M^n be a flat submanifold of S^{2n-1} which is Lagrangian in $\mathbb{R}^{2n}$, and $\pi : S^{2n-1} \to \mathbb{C}P^{n-1}$ be the Hopf fibration. Then $M = \pi^{-1}(\pi(M))$ and $\pi(M)$ is a flat Lagrangian submanifold of $\mathbb{C}P^{n-1}$. (See [Te02] for a simple geometric proof.)

Thus, we have the following:

Space of spherical Egorov metrics

$\cong$ Space of flat submanifolds of S^{2n-1} that is Lagrangian in $\mathbb{R}^{2n}$ modulo $\mathrm{U}(n)$

$\cong$ Space of local flat Lagrangian submanifolds in $\mathbb{C}P^{n-1}$ modulo $\mathrm{U}(n)$

4. Dressing action, loop group factorizatio

Let $G = \mathrm{GL}(n, \mathbb{C})$. For $\epsilon > 0$, let $\mathcal{O}_\epsilon = \{\lambda \in \mathbb{C} \mid |\lambda| < \epsilon\}$, $\mathcal{O}_{1/\epsilon} = \{\lambda \in \mathbb{C} \cup \{\infty\} \mid |\lambda| > 1/\epsilon\}$. Henceforth we will use the following loop groups:

$\Lambda(G) = \{$ holomorphic map from $\mathbb{C} \cap \mathcal{O}_{1/\epsilon}$ to G $\}$,

$\Lambda_+(G) = \{$ holomorphic map from $\mathbb{C}$ to G $\}$,

$\Lambda_-(G) = \{$ holomorphic map f from $\mathcal{O}_{1/\epsilon}$ to G with $f(\infty) = e$ $\}$.

Birkhoff Factorization Theorem The multiplication maps from $\Lambda_+ G \times \Lambda_- G$ and $\Lambda_- G \times \Lambda_+ G$ to $\Lambda(G)$ are $1-1$ and the images are open and dense. In particular, there exists an open dense subset $\Lambda(G)_0$ of $\Lambda(G)$ such that given $g \in \Lambda(G)_0$, g can be factored uniquely as $g = g_+ g_- = h_- h_+$ with $g_+, h_+ \in \Lambda_+(G)$ and $g_-, h_- \in \Lambda_-(G)$.

Using the involutions τ, σ associated to the symmetric space $\mathrm{U}(n)/\mathrm{O}(n)$ as before, we will denote

$$\begin{aligned}
\Lambda^{\tau}(G) &= \{f \in \Lambda(G) \mid \ \tau(f(\lambda)) = f(\bar{\lambda})\}, \\
\Lambda^{\tau,\sigma}(G) &= \{f \in \Lambda^{\tau}(G) \mid \ \sigma(f(\lambda)) = f(-\lambda)\}, \\
\Lambda^{\tau}_{\pm}(G) &= \Lambda^{\tau}(G) \cap \Lambda_{\pm}(G), \\
\Lambda^{\tau,\sigma}_{\pm}(G) &= \Lambda^{\tau,\sigma}(G) \cap \Lambda_{\pm}(G).
\end{aligned}$$

Corollary Suppose $g \in \Lambda(G)$ is factored as $g = g_+ g_-$ with $g_+ \in \Lambda_+(G)$ and $g_- \in \Lambda_-(G)$. If $\tau\sigma = \sigma\tau$, then

(i) $g \in \Lambda^{\tau}(G)$ implies that $g_{\pm} \in \Lambda^{\tau}_{\pm}(G)$,

(ii) $g \in \Lambda^{\tau,\sigma}(G)$ implies that $g_{\pm} \in \Lambda^{\tau,\sigma}_{\pm}(G)$.

We need the dressing action of Zakharov and Shabat [ZS79]. Suppose G_+, G_- are subgroups of a Lie group G and the multiplication map from $G_+ \times G_-$ to G is a bijection. Then every $g \in G$ can be factored uniquely as $g = g_+ g_-$ with $g_+ \in G_+$ and $g_- \in G_-$. Moreover, the space of right cosets G/G_- can be identified with G_+, so the canonical action of G_- on G/G_- by left multiplication, $g_- \cdot (gG_-) = g_- g G_-$, induces an action $*$ of G_- on G_+. The action $*$ is called the *dressing action*. The dressing action can be computed by factorization. In fact, $g_- * g_+ = \tilde{g_+}$, where $g_- g_+ = \tilde{g_+}\tilde{g_-}$ with $\tilde{g_+} \in G_+$ and $\tilde{g_-} \in G_-$.
If the multiplication map from $G_+ \times G_-$ to G is one-to-one but only onto an open, dense subset of G, then the dressing actions are defined on an open neighborhood of the identity e in $G_\pm$.

Terng and Uhlenbeck developed Bäcklund theory of the $\mathrm{U}(n)/\mathrm{O}(n)$-system in [TU00] by applying the dressing action of simple rational elements.
Let π be the Hermitian projection of $\mathbb{C}^n$ onto V and $z \in \mathbb{C} \setminus \mathbb{R}$. Then the set consisting of

$$g_{z,\pi}(\lambda) = \pi + \frac{\lambda - z}{\lambda - \bar{z}}\pi^{\perp}$$

generates $R\Lambda_{-}^{\tau}(G)$ by Uhlenbeck's Theorem [Uh89]. One can then verify $g_{\mathrm{i}\alpha,\pi} \in \Lambda_{-}^{\tau,\sigma}(G)$.

By Birkhoff Factorization Theorem, the dressing action of $\Lambda_{-}(G)$ on $\Lambda_{+}(G)$ is only defined locally.

However, Terng and Uhlenbeck have shown that the $\mathrm{U}(n)$-reality condition implies that the simple elements act on $\Lambda_+^\tau(G)$ globally and explicitly. Since simple elements generate $R\Lambda_-^\tau(G)$, the group $R\Lambda_-^\tau(G)$ acts globally on $\Lambda_+^\tau(G)$.

Lemma: [TU00] Let $z \in \mathbb{C}$, π a Hermitian projection of $\mathbb{C}^n$ onto V, $g_{z,\pi}$ a simple element of $R\Lambda_-^\tau(G)$ and $f \in \Lambda_+^\tau(G)$. Then $g_{z,\pi} f$ can always be factored uniquely as

$$g_{z,\pi} f = \tilde{f} g_{z,\tilde{\pi}} \in \Lambda_+^\tau(G) \times R\Lambda_-^\tau(G),$$

where $\tilde{\pi}$ is the Hermitian projection onto $f(\bar{z})^{-1}(V)$.

Theorem: [TU00] The group $\mathbb{R}^* \ltimes R\Lambda_-^{\tau,\sigma}(G)$ acts on the space $\mathcal{M}$ of solutions of the $\mathrm{U}(n)/\mathrm{O}(n)$-system.
Here $(r * v)(x) = r^{-1}v(r^{-1}x)$ for $r \in R^*$,
and $g_{\mathrm{i}\alpha,\pi} * F = F + 2\alpha(\tilde{\pi})_*$,
where $\tilde{\pi}(x)$ is the Hermitian projection onto the linear subspace $E(x, i\alpha)^*(V)$, and V is the image of the projection π. The multiplication in $\mathbb{R}^* \ltimes \Lambda_-^{\tau,\sigma}(G)$ is defined by

$$(r_1, g_1) \cdot (r_2, g_2) = (r_1 r_2, g_1(\rho(r_1)(g_2))),$$

where $\rho(r)(g)(\lambda) = g(r\lambda)$.

Example: When π is a Hermitian projection onto 1-dimensional subspace $\mathbb{C} \cdot \ell$. Then $\gamma = E(i\alpha)^*\ell$ can be solved uniquely from:

$$\begin{cases} (\gamma_i)_{x_j} = f_{ij}\gamma_j, & i \neq j, \\ (\gamma_i)_{x_i} = -\alpha\gamma_i - \sum_j f_{ij}\gamma_j, \\ \gamma_i(0, \cdots, 0) = \ell_i. \end{cases}$$

Then $\tilde{F} = g_{\mathrm{i}\alpha,\pi} * F$ is given by

$$\tilde{f_{ij}} = f_{ij} + \frac{2\alpha\gamma_i\gamma_j}{\sum_k \gamma_k^2}.$$

To extend the theory to flat Lagrangian submanifolds in $\mathbb{C}^n$, one has to enlarge the system to contain the Egorov metric.

Extension: The extended Lax n-tuple is naturally the following: $\theta_\lambda = \begin{pmatrix} \mathrm{i}\lambda\delta+[\delta,F] & \delta b \\ 0 & 0 \end{pmatrix}$.

The corresponding loop group is $\Lambda(G)$ for $G = \left\{ W = \begin{pmatrix} E & v \\ 0 & s \end{pmatrix} \mid v \in \mathbb{C}^n, s \in \mathbb{C}^* \right\}$.

Let $h = \hat{\pi} + \frac{\lambda-\mathrm{i}\alpha}{\lambda+\mathrm{i}\alpha}(I - \hat{\pi}) = \begin{pmatrix} g_{i\alpha,\pi} & 0 \\ 0 & \frac{\lambda-\mathrm{i}\alpha}{\lambda+\mathrm{i}\alpha} \end{pmatrix}$ where $\hat{\pi}$ denotes the Hermitian projection of $\mathbb{C}^{n+1}$ onto $\hat{\ell} = \begin{pmatrix} \ell \\ 0 \end{pmatrix}$. Denote $\tilde{h} = h_{-\mathrm{i}\alpha,\mathrm{i}\alpha,\tilde{\hat{\pi}}} = \begin{pmatrix} g_{\mathrm{i}\alpha,\tilde{\pi}} & \xi \\ 0 & \frac{\lambda-\mathrm{i}\alpha}{\lambda+\mathrm{i}\alpha} \end{pmatrix}$.

Main Theorem 1: Given a flat Lagrangian submanifold X in $\mathbb{C}^n$ with frame W and the potential ϕ for its Egorov metric, the dressing action of $h_{-i\alpha,i\alpha,\hat{\pi}}$ on W can be solved explicitly as follows: $h(\lambda)W(x,\lambda) = \tilde{W}(x,\lambda)\tilde{h}(x,\lambda)$ where $\tilde{h} = \tilde{\hat{\pi}} + \frac{\lambda - \mathrm{i}\alpha}{\lambda + \mathrm{i}\alpha}(I - \tilde{\hat{\pi}})$ and $\tilde{\hat{\pi}}$ is the projection with respect to the following decomposition:
$\mathbb{C}^{n+1} = W(x,-i\alpha)^{-1}\hat{\ell} \,\oplus\, W(x,i\alpha)^{-1}\hat{\ell}^{\perp}$.

- $\xi = \frac{2\mathrm{i}\alpha\varphi}{\lambda+\mathrm{i}\alpha} \cdot \frac{\gamma}{|\gamma|^2}$, where $\varphi = \ell^t X(x, \mathrm{i}\alpha)$;
- $\tilde{\phi} = \phi + \frac{2\alpha\varphi^2}{|\gamma|^2}$;
- A new flat Lagrangian submanifold by $\hat{X} := g_{\mathrm{i}\alpha,\pi}^{-1}\tilde{X} = X(x,\lambda) - \frac{2\mathrm{i}\alpha\varphi}{\lambda+\mathrm{i}\alpha} \cdot \frac{E(x,\lambda)\gamma}{|\gamma|^2}$.

Dajczer and Tojeiro generalized Ribaucour transformations in [DT00, 02] by geometric methods. In their formulas one has to solve $\varphi_{x_i} = b_i\gamma_i$ to obtain φ. Thus there are two advantages of our formula.

Main Theorem 2: The dressing action of $h_{-\mathrm{i}\alpha,\mathrm{i}\alpha,\hat{\pi}}$ on any given flat Lagrangian submanifold in $S^{2n-1} \subset \mathbb{C}^n$ can be solved explicitly as follows ($\gamma = E(x,\mathrm{i}\alpha)^*\ell$):

- A new flat Lagrangian submanifold is given by $\hat{X} := g_{\mathrm{i}\alpha,\pi}^{-1}\tilde{X} = X(x,\lambda) + \frac{2\mathrm{i}\gamma^t b}{(\lambda+\mathrm{i}\alpha)|\gamma|^2} \cdot E(x,\lambda)\gamma$;
- The Egorov metric for $\hat{X}$ (or $\tilde{X}$) is given by $\mathrm{d}s^2 = \sum_i \tilde{b}_i^2\,\mathrm{d}x_i^2 = \sum_i \tilde{\phi}_{x_i}\,\mathrm{d}x_i^2$, where $\tilde{\phi} = \phi + \frac{2(\gamma^t b)^2}{\alpha|\gamma|^2}$.

Example: “Vacuum” flat Lagrangian submanifold in $\mathbb{C}^n$:
$F \equiv 0$ or $f_{ij} \equiv 0$, and $E = \exp(\sum_j \mathrm{i}\lambda x_j e_{jj})$;
the immersion X_0 can be represented as direct products of n plane curves:

$$X_0 = z_1 \times \cdots \times z_n : I_1 \times \cdots \times I_n \to \mathbb{C} \times \cdots \times \mathbb{C} = \mathbb{C}^n,$$

with each I_i being an open interval in $\mathbb{R}$. Every plane curve z_i has nowhere vanishing curvature $k_i(x_i) = \lambda / b_i(x_i)$, thus normal bundle is non-degenerate. Here $z_j(x_j) = \int b_j(x_j) e^{\mathrm{i}\lambda x_j}\,\mathrm{d}x_j$. The potential of the Egorov metric is $\phi_0 = \sum_j \int b_j(x_j)^2\,\mathrm{d}x_j$.

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“Vacuum” solution in $\mathbb{C}P^{n-1}$: is the Clifford torus with radii r_i. The potential is $\tilde{\phi}_0 = \sum_j r_j^2 x_j$.

Reduction:

- An important observation in [TU00]: Let $V_0 = V_1 \oplus V_2$ and $V_1 \perp V_2$. Let π_i denote the Hermitian projection to V_i. Then:

$$g_{z,\pi_0} = \frac{\lambda - \bar{z}}{\lambda - z} g_{z,\pi_1} g_{z,\pi_2}.$$

  So we only need to take care of projections into one dimensional subspace when computing the dressing actions of simple elements.

- We classify the rational elements in $\Lambda_-^{\tau,\sigma}(G)$ with two simple poles by Uhlenbeck's Theorem. The dressing action of the rational element g with two simple poles reduces to the dressing action of a rational element with one simple pole.

Example: “one-soliton” flat Lagrangian submanifold in $\mathbb{C}^n$:

$$X_1 = X_0 - \frac{2\alpha(\alpha+\mathrm{i}\lambda)\varphi}{(\lambda^2+\alpha^2)|\gamma|^2} \cdot (e^{\mathrm{i}\lambda x_1}\gamma_1, \cdots, e^{\mathrm{i}\lambda x_n}\gamma_n)^t,$$

$$\mathrm{d}s_1^2 = \sum_j (\phi_1)_{x_j}\,\mathrm{d}x_j^2, \quad \phi_1 = \phi_0 + \frac{2\alpha\varphi^2}{|\gamma|^2}.$$ where

$\varphi = \sum_j \int \ell_j b_j e^{-\alpha x_j}\,\mathrm{d}x_j$ and $|\gamma|^2 = \sum_j \ell_j^2 e^{-2\alpha x_j}$.

“one-soliton” flat Lagrangian submanifold in $\mathbb{C}P^{n-1}$:

$$\tilde{X}_1 = X_0 + \frac{2(\alpha+\mathrm{i}\lambda)\sum_j \ell_j r_j e^{-\alpha x_j}}{(\lambda^2+\alpha^2)|\gamma|^2}(e^{\mathrm{i}\lambda x_1}\gamma_1, \cdots, e^{\mathrm{i}\lambda x_n}\gamma_n)^t,$$

$$\mathrm{d}\tilde{s}_1^2 = \sum_j (\tilde{\phi}_1)_{x_j}\,\mathrm{d}x_j^2, \quad \tilde{\phi}_1 = \sum_j r_j^2 x_j + \frac{2(\sum_j \ell_j r_j e^{-\alpha x_j})^2}{\alpha\sum_j \ell_j^2 e^{-2\alpha x_j}}.$$

§5. A program for submanifold geometries

Main interest: Find special submanifolds which admit many deformations and explicit solutions
Classical examples: Surfaces with constant negative Gaussian curvature or constant mean curvature (including minimal surface)
The Gauss-Codazzi equations for these surfaces are *integrable systems* !
Moreover, the classical geometrical transformations of Bäcklund, Darboux and Ribaucour can be constructed by *dressing actions* through loop group factorization [TU00].
Question: How to generalize and find interesting submanifolds in other space?

ZS-AKNS construction Systematic construction from a complex semi-simple Lie algebra and finite order automorphisms. Famous examples includes:

- KdV equation,
- Sine-Gordon equation,
- non-linear Schrödinger equation,
- n-wave equation,
- the equation for harmonic maps from the plane to a compact Lie group.

Example: $\mathcal{G} = \mathrm{sl}(3, \mathbb{C})$. (Type A_2)

- Tzitzéica equation: $\omega_{xy} = e^{\omega} - e^{-2\omega}$ (Indefinite affine spheres in $\mathbb{R}^3$); [BS99, BE00, Wa03]
- $\omega_{z\bar{z}} = -e^{\omega} - e^{-2\omega}$ (Definite affine spheres in $\mathbb{R}^3$);
- $\omega_{z\bar{z}} = -e^{\omega} + e^{-2\omega}$ (Special Lagrangian cones in $\mathbb{C}^3$); [Ha00, Ma-Ma01, Mc03, HTU, etc.]
- structure equations for minimal surfaces in $\mathbb{C}P^2$;
- structure equations for Hamiltonian stationary Lagrangian surfaces in $\mathbb{C}P^2$. [He-Ro00]

$\vdots$

Direct approach: *Can we identify geometric objects corresponding to the various integrable systems associated to general simple Lie algebras?*

This program was first proposed by Terng in [Te97] *Soliton equations and differential geometry*, and was then carried out for the real Grassmannian system in [BDPT00]

Our discussion above is thus another example of this program regarding the $\mathrm{U}(n)/\mathrm{O}(n)$-system and one of its extensions.

The key link between these integrable systems and submanifold geometries is the one parameter family of some Lie algebra valued flat connection one-form: θ_λ, i.e. Lax n-tuple.

For more examples and deeper results, please see the survey math.DG/0212372.