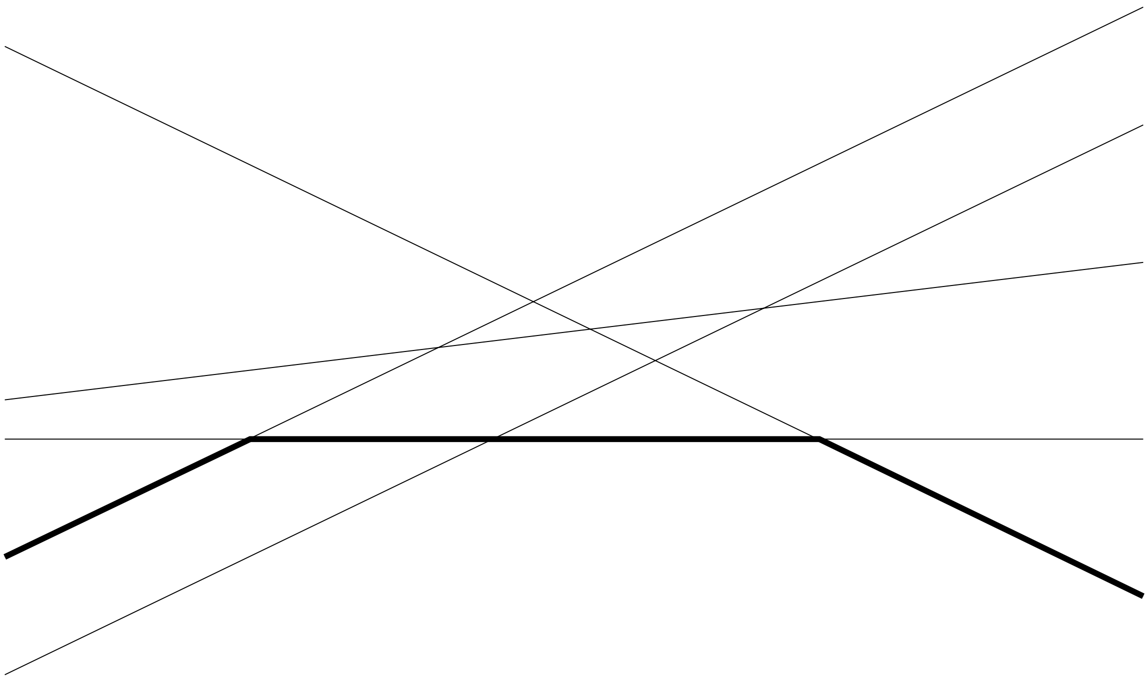
Monotone paths in line arrangements with a small number of directions

Adrian Dumitrescu

University of Wisconsin–Milwaukee

Arrangement of lines



Arrangement of five lines.

No vertical line.

A monotone path of length three.

Length is number of turns plus one (i.e., the number of edges of the path).

Background

λ_n = the maximum possible length of an x-monotone polygonal line in an arrangement of n lines (over all arrangements).

The problem to estimate λ_n was posed in [Edelsbrunner and Guibas, 1989]

An application of this problem can be found in [Yamamoto et. al., 1989]

Related problem: the k-level (or its dual, the k-set) problem in the plane. The k-level is the closure of the set of points of the lines with the property that there are exactly k lines below them. Estimate the maximum length of the k-level (over all arrangements).
Note: the k-level is a monotone path which turns at each vertex on the path.

Previous results

Lower bounds

[Sharir, 1987] $\Omega(n^{3/2})$

[Matoušek, 1991] $\Omega(n^{5/3})$

[Radoičić and Tóth, 2001] $\Omega(n^{7/4})$

[Balogh et. al., 2003] $\Omega(n^2/C^{\sqrt{\log n}})$, $C > 1$.

Upper bounds

[Radoičić and Tóth, 2001] $\lambda_n < 5n^2/12$.

Related problem: arrangements of pseudolines.
[Matoušek, 1991] $\Omega(n^2/\log n)$ lower bound.

Theorem 1 *Let $L_k(n)$ be the maximum length of a monotone path in an n-line arrangement whose lines have at most k distinct slopes. Then*

(i) $L_1(n) = 1$.

(ii) $L_2(n) = n$.

(iii) $2n - O(\sqrt{n}) \leq L_3(n) \leq 2n + 1$.

(iv) $L_4(n) = \Theta(n^{3/2})$.

(v) $L_5(n) = \Theta(n^{5/3})$.

(vi) $L_6(n) = O(n^{9/5})$.

(vii) $L_7(n) = O(n^{15/8})$.

(viii) For any $k \geq 4$, $L_k(n) \leq 25 \cdot k \cdot n^{2-\frac{1}{F_{k-2}}}$, *where F_k is the k-th Fibonacci number.*

The Fibonacci numbers are defined by the recurrence: $F_0 = 1$, $F_1 = 1$, $F_i = F_{i-1} + F_{i-2}$, for $i \geq 2$.

Fibonacci numbers

$F_0 = 1$

$F_1 = 1$

$F_2 = 2$; $L_4(n) = \Theta(n^{2-1/F_2}) = \Theta(n^{3/2})$.

$F_3 = 3$; $L_5(n) = \Theta(n^{2-1/F_3}) = \Theta(n^{5/3})$.

$F_4 = 5$; $L_6(n) = O(n^{2-1/F_4}) = O(n^{9/5})$.

$F_5 = 8$

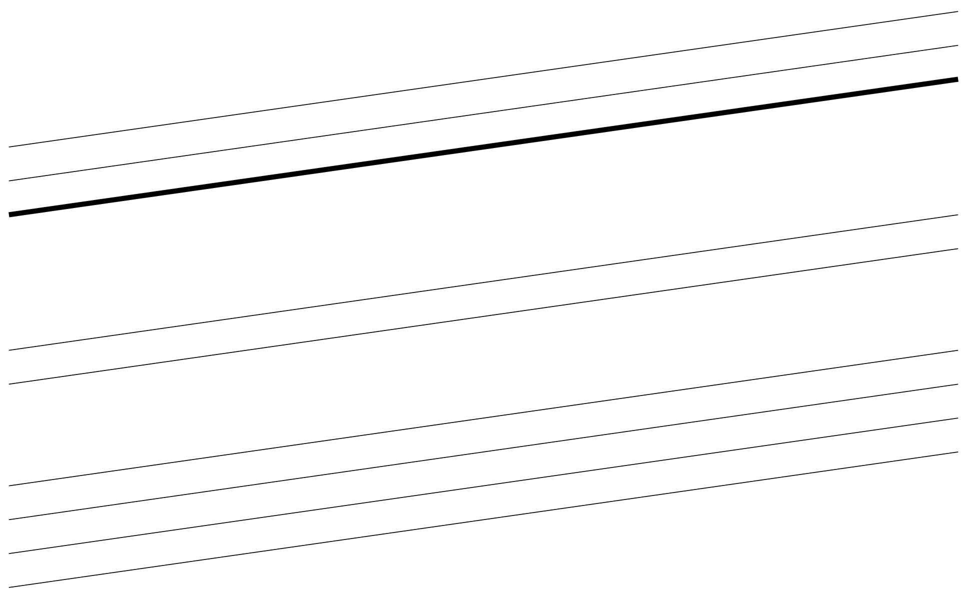
$F_6 = 13$

$F_7 = 21$

$F_8 = 34$

$\vdots$

One slope

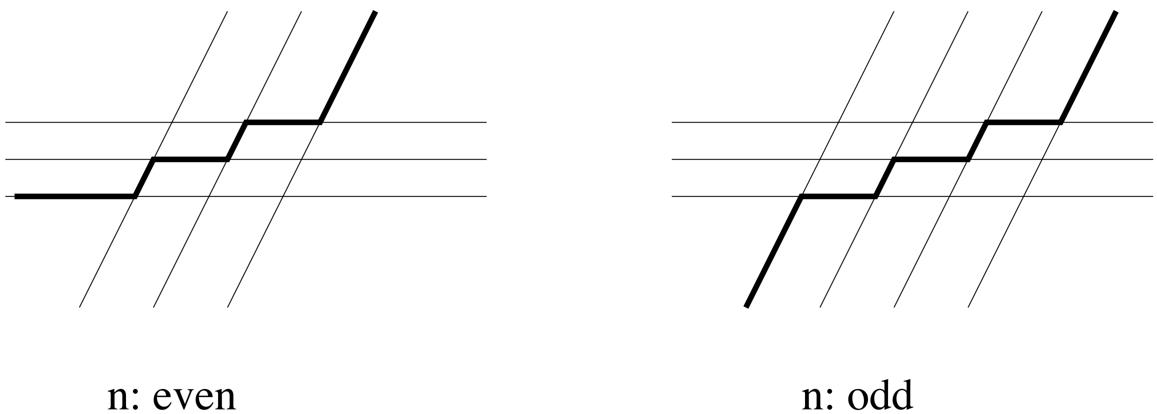


Arrangement of parallel lines.

$L_1(n) = 1$

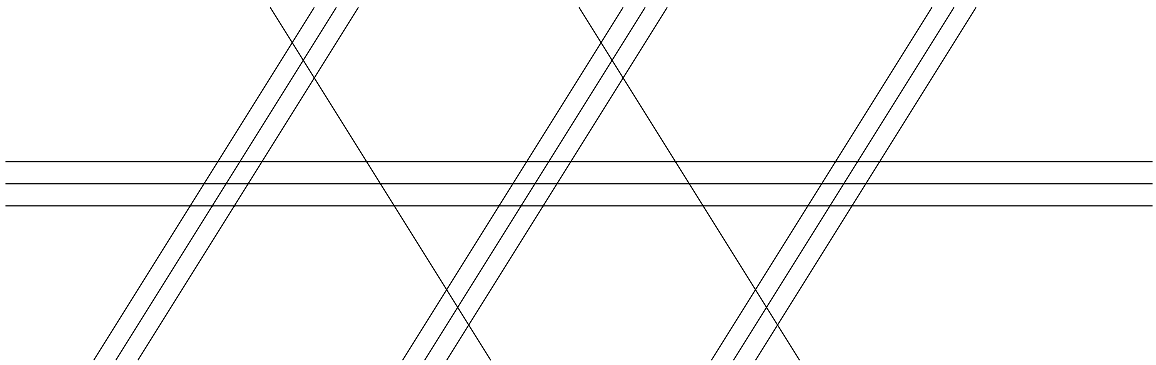
Having more lines does not help!

Two slopes



Arrangement of lines with two slopes which admits a monotone path of length n.

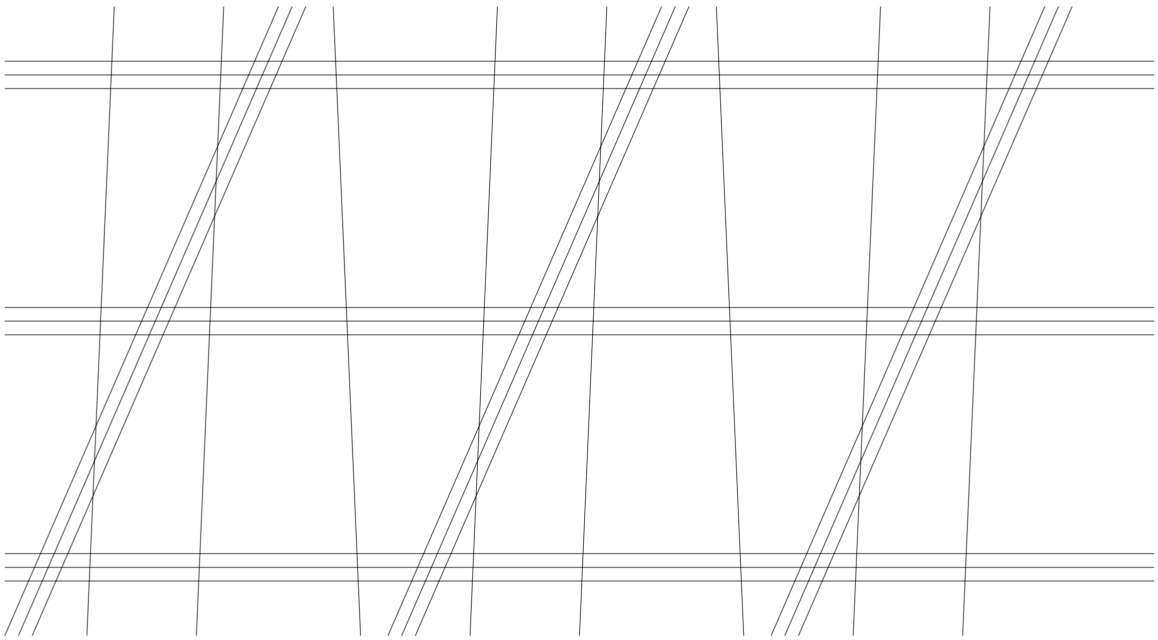
Three slopes



Arrangement of lines with three slopes which admits a monotone path of length $2n - O(\sqrt{n})$. It consists of: a bundle of m horizontal lines, m bundles of m lines each, having slope of 1, and $m - 1$ lines of negative slope (say -1).

$n = m^2 + 2m - 1$; monotone path of length $2m^2 + m - 1$; $m = 3$ in this example.

Four slopes



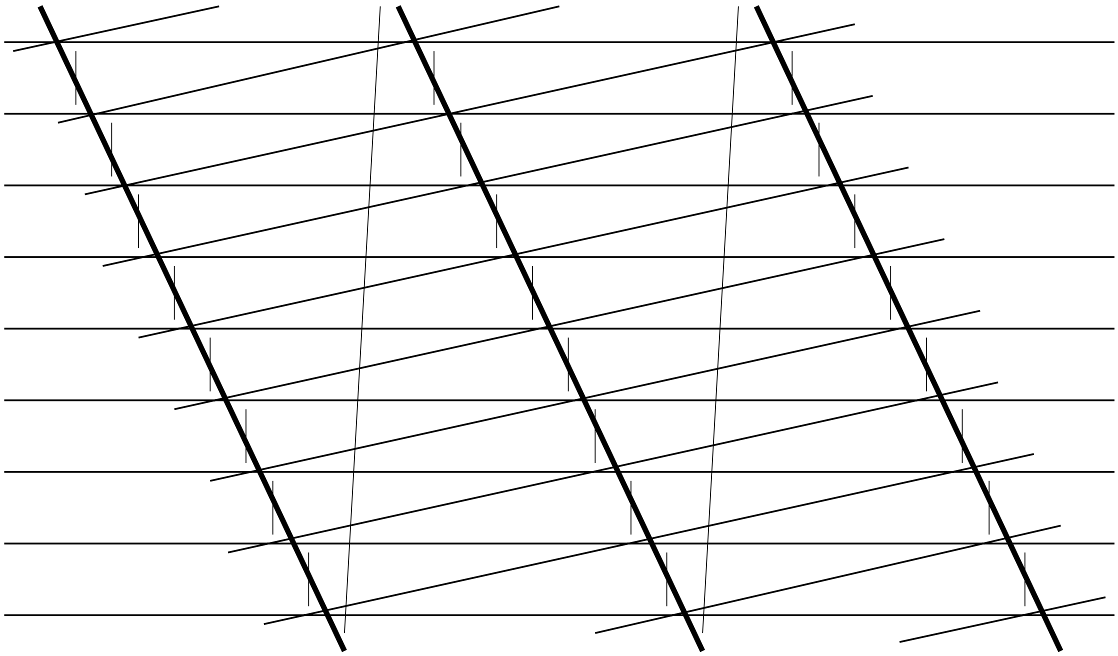
Arrangement of lines with four slopes which admits a monotone path of length $\Omega(n^{3/2})$. It consists of: m bundles of m horizontal lines each; m bundles of m lines each at (say) 60°; $m(m-1)$ near vertical parallel lines of positive slope, and $m-1$ near vertical parallel lines of negative slope.

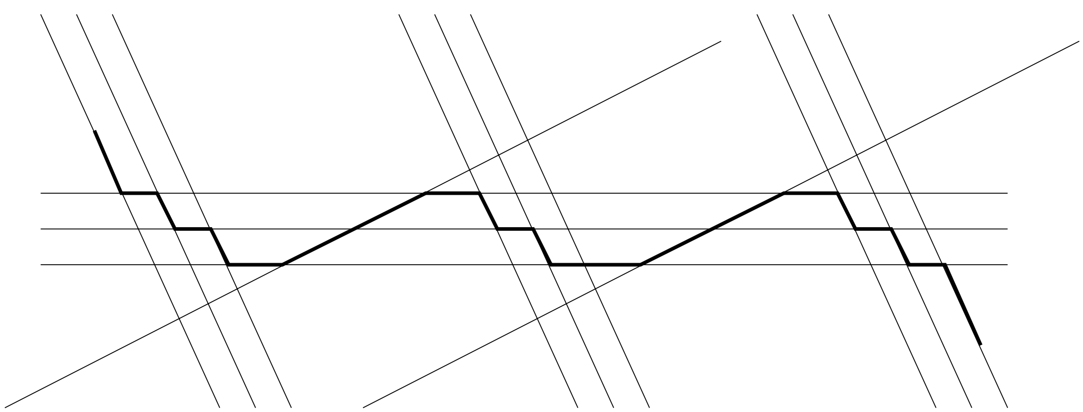
$n = \Theta(m^2)$; monotone path of length $\Omega(m^3)$; $m = 3$ in this example.

Five slopes

Arrangement of lines with four slopes which admits a monotone path of length $\Omega(n^{5/3})$. It consists of: m^2 bundles of m horizontal lines each; m bundles of m^2 lines each (descending); m^2+m-1 bundles of $m-1$ lines each (ascending); $m(m^2-1)$ nearly vertical lines of negative slope; $m-1$ nearly vertical lines of positive slope.

$n = \Theta(m^3)$; monotone path of length $\Omega(m^5)$; $m = 3$ in this example.





Notation

p = monotone path
$l(p)$ = the length of p
$t(p)$ = the number of turns of p =$l(p) - 1$.
slopes: $1, 2, \ldots, k$.

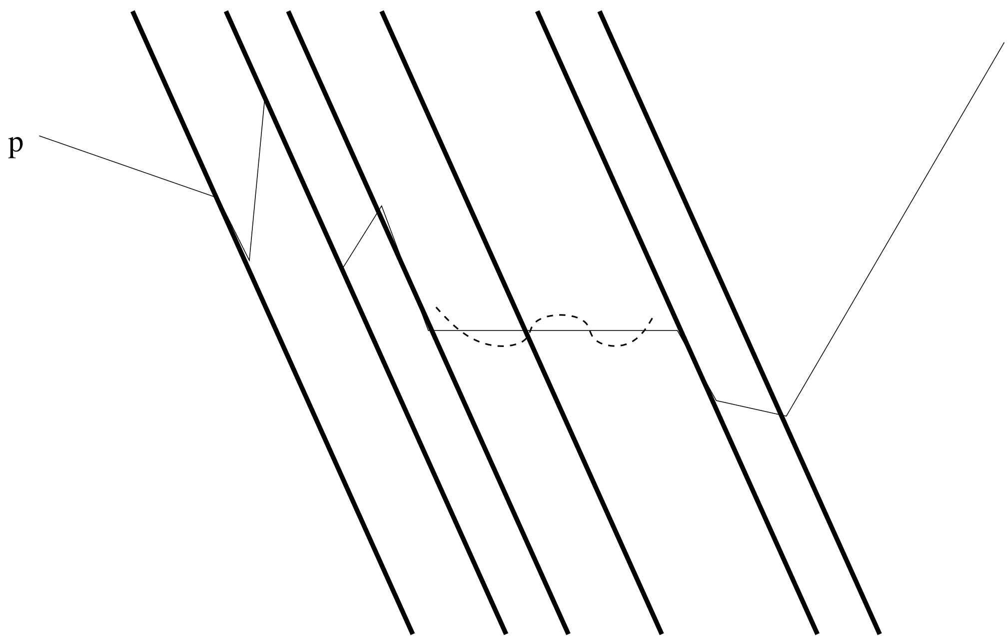
Consider $\mathcal{A}_1$.
$Q(\mathcal{A}_1, p)$ = the set of cells of $\mathcal{A}_1$ which are visited by p and in which p turns.
l_c = the length of the portion of p inside a cell $c \in Q(\mathcal{A}_1, p)$.

p' = *monotone shortcut path* in $\mathcal{A}_1$ (or $\mathcal{A}_{1,k}$).

Similar:
Consider $\mathcal{A}_{1,k}$.
A cell of $\mathcal{A}_{1,k}$ is said to be *visited* by p if p intersects its interior.
$Q(\mathcal{A}_{1,k}, p)$ = the set of cells of $\mathcal{A}_{1,k}$ which are visited by p and in which p turns.
l_c = the length of the portion of p inside a cell $c \in Q(\mathcal{A}_{1,k}, p)$.

Proof of Theorem 1

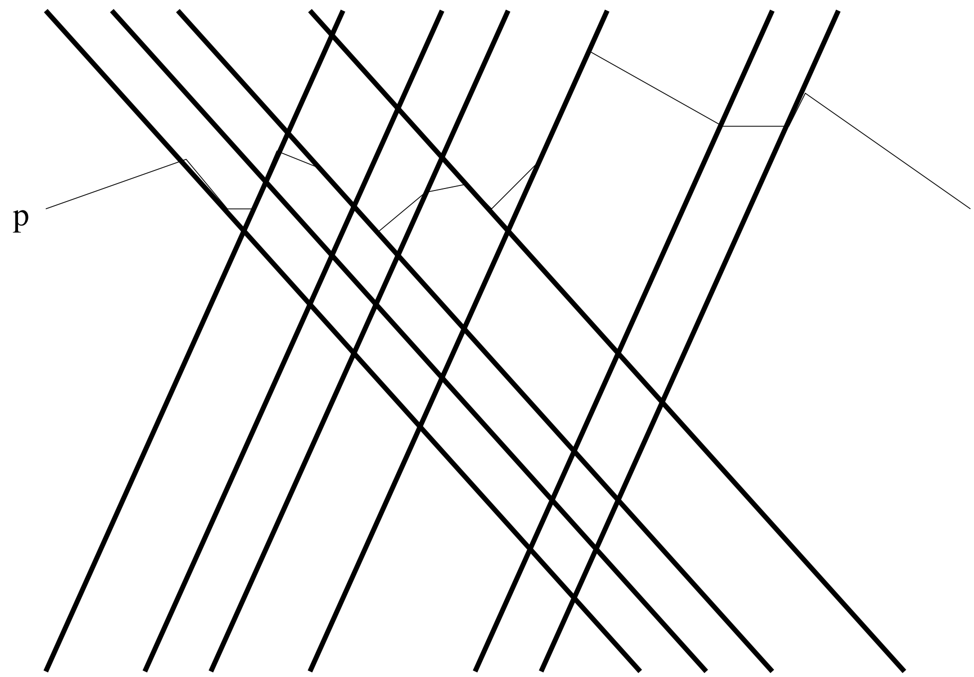
By induction on k.



Arrangement $\mathcal{A}_1$ of parallel (thick) lines of minimum slope; none of these can be revisited by p.

$$l(p) \leq l(p') + \sum_{c \in Q(\mathcal{A}_1, p)} (l_c - 1). \qquad (1)$$

$$t(p) \leq t(p') + \sum_{c \in Q(\mathcal{A}_1, p)} t_c.$$



Arrangement $\mathcal{A}_{1,k}$ of lines of minimum and maximum slope; none of these can be revisited by p.

$$l(p) \leq l(p') + \sum_{c \in Q(\mathcal{A}_{1,k}, p)} (l_c - 1). \qquad (2)$$

$$t(p) \leq t(p') + \sum_{c \in Q(\mathcal{A}_{1,k}, p)} t_c.$$

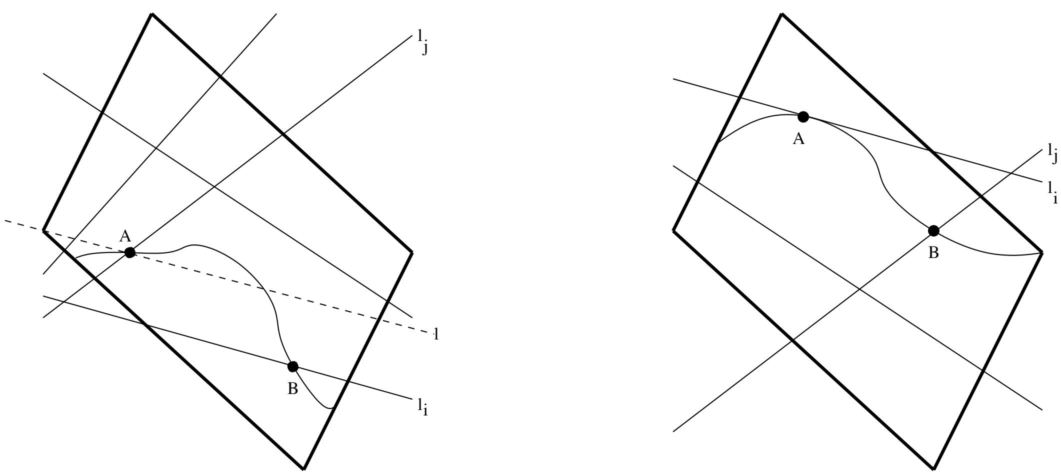
Lemma 1 *Let* p' *and* p'' *be shortcut monotone paths in the arrangements* $\mathcal{A}_1$ *and* $\mathcal{A}_{1,k}$*, respectively. Put* $q' = |Q(\mathcal{A}_1, p')|$ *and* $q'' = |Q(\mathcal{A}_{1,k}, p'')|$*. Then*

(i) $l(p') \leq 2n_1 + 1$ *and* $q' \leq n_1 + 1$.

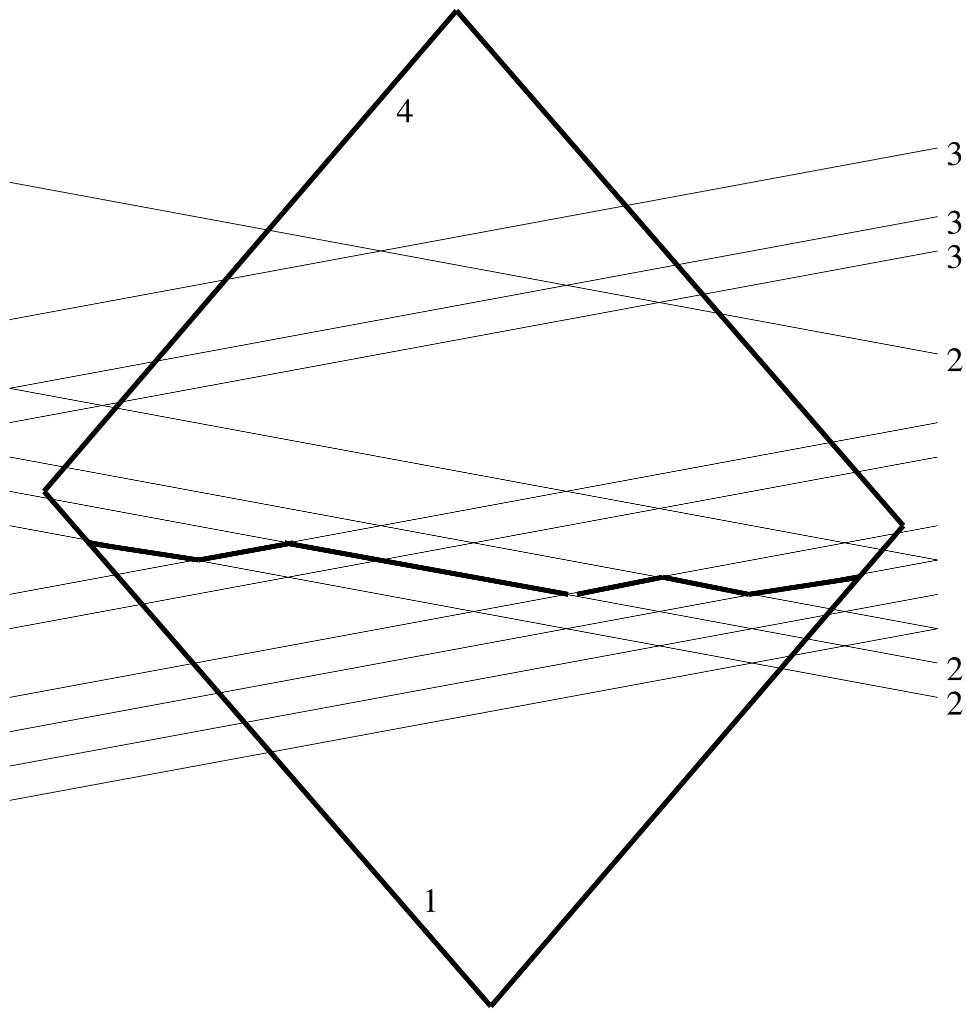
(ii) $l(p'') \leq 2n_1 + 2n_k + 1$ *and* $q'' \leq n_1 + n_k + 1$.

Or:
$t(p') \leq 2n_1$
$t(p'') \leq 2n_1 + 2n_k$

Lemma 2 *Consider an arrangement* $\mathcal{A} = \mathcal{A}_{1,\ldots,k}$ *of* n *lines having* k *distinct slopes* ($k \geq 4$)*, and let* p *be a monotone path in* $\mathcal{A}$*. Let* c *be a convex cell* $c \in Q(\mathcal{A}_{1,k}, p)$*, and let* p_c *be the portion of* p *which lies in the interior of* c*. Assume that* ℓ_i *and* ℓ_j *are two lines of minimum and maximum slope respectively, which intersect* p_c*. Then* ℓ_i *and* ℓ_j *intersect in the interior of* c*.*



A cell $c \in Q(\mathcal{A}_{1,\ldots,k}, p)$, and the portion p_c of p which lies in the interior of c.



A cell $c \in Q(\mathcal{A}_{1,4}, p)$, and the portion p_c of p which lies in the interior of c. All the $3 \times 4 = 12$ vertices of the arrangement of the lines in $L_2 \cup L_3$ are in the interior of c.

Lemma 3 *Let $m \geq 1$ be the number of vertices in a line arrangement $\mathcal{A}$ of n lines having two distinct slopes, and let p be a monotone path in $\mathcal{A}$. Then $l(p) \leq \min(n,\ 2\sqrt{m}+1)$. Further on, consider an arrangement of n lines having four distinct slopes, a convex cell $c \in Q(\mathcal{A}_{1,4}, p)$, and the portion p_c of p which lies in the interior of c. Then $l(p_c) \leq 2\sqrt{m_c}+1 \leq 3\sqrt{m_c}$, where m_c is the number of vertices of $\mathcal{A}_{2,3}$ in the interior of c.*

General idea: bound the length in terms of the number of vertices.

Proof of Lemma 3

First part:
$n = n_1 + n_2$, w.l.o.g. $n_1 \le n_2$.
$n \ge 2n_1$ and $m = n_1 n_2 \ge n_1^2$.
$t(p) \le 2n_1 \le n$.
$l(p) \le 2n_1 + 1 \le n + 1$.
In fact, $l(p) \le n$.
Also, $n_1 \le \sqrt{m}$, so
$l(p) \le 2n_1 + 1 \le 2\sqrt{m} + 1$.

Second part:
$p_c =$ contiguous portion of a monotone path in $\mathcal{A}_{2,3}$ which lies in c.

let L_2 (resp. L_3) be the set lines of slope 2 (resp. 3) which intersect p_c. Since $c \in Q(\mathcal{A}_{1,4}, p)$, $|L_2|, |L_3| \ge 1$.

By the convexity of c and the ordering of the slopes (Lemma 2 for $k = 4$), all the $|L_2| \cdot |L_3|$ vertices of the arrangement of the lines in $L_2 \cup L_3$ are in the interior of c. Thus $m_c \ge |L_2| \cdot |L_3|$. Assuming $|L_2| \le |L_3|$, we have $m_c \ge |L_2|^2$. Thus $l(p_c) \le 2|L_2| + 1 \le 2\sqrt{m_c} + 1 \le 3\sqrt{m_c}$.

Corollary 1 $L_4(n) = O(n^{2-\frac{1}{F_2}}) = O(n^{3/2})$.

Proof.
Consider $\mathcal{A}_{1,4}$ and use (2). By Lemma 1(ii),

$l(p') \leq 2n_1 + 2n_4 + 1 \leq 2n + 1$, and

$q = |Q(\mathcal{A}_{1,4}, p)| \leq n_1 + n_4 + 1 \leq n + 1$.

$$l(p) \leq 2n + 1 + \sum_{c \in Q(\mathcal{A}_{1,4},p)} (l_c - 1)$$
$$\leq 2n + 1 + 2 \sum_{c \in Q(\mathcal{A}_{1,4},p)} \sqrt{m_c}.$$

By Jensen's inequality

$$l(p) \leq 2n + 1 + 2q\sqrt{\frac{\binom{n}{2}}{q}}$$
$$\leq 2n + 1 + 2\sqrt{n+1}\sqrt{\binom{n}{2}} = O(n^{3/2}).$$

□

Lemma 4 *Let* $m \geq 1$ *be the number of vertices in a line arrangement* $\mathcal{A}$ *of* n *lines having three distinct slopes, and let* p *be a monotone path in* $\mathcal{A}$*. Then* $l(p) \leq \min(2n_1 + 2n_3 + 1, 6m^{2/3})$*. Further on, consider an arrangement of* n *lines having five distinct slopes, a convex cell* $c \in Q(\mathcal{A}_{1,5}, p)$*, and the portion* p_c *of* p *which lies in the interior of* c*. Then* $l(p_c) \leq 6m_c^{2/3}$*, where* m_c *is the number of vertices of* $\mathcal{A}_{2,3,4}$ *in the interior of* c*.*

Obs. More slopes give a weaker bound.

Lemma 5 *Let $k \geq 2$. Let $m \geq 1$ be the number of vertices in a line arrangement $\mathcal{A}$ of n lines having k distinct slopes, and let p be a monotone path in $\mathcal{A}$. Then*

$$l(p) \leq c_k \cdot m^{1-\frac{1}{F_k}},$$

where

$$c_k = 5 \cdot k \cdot 3^{\sum_{i=2}^{k-1} \frac{1}{F_i}},$$

and F_k is the k-th Fibonacci number. Further on, consider an arrangement of n lines having $k+2$ distinct slopes, a convex cell $c \in Q(\mathcal{A}_{1,k+2}, p)$, and the portion p_c of p which lies in the interior of c. Then

$$l(p_c) \leq c_k \cdot m_c^{1-\frac{1}{F_k}},$$

where m_c is the number of vertices of $\mathcal{A}_{2,\dots,k+1}$ in the interior of c.

Corollary 2 *For any* $k \geq 4$,
$L_k(n) \leq 25 \cdot k \cdot n^{2-\frac{1}{F_{k-2}}}$, *where* F_k *is the* k*-th Fibonacci number.*

Proof.

Let $m \leq \binom{n}{2}$ be the number of vertices of $\mathcal{A}$, and p be a monotone path in $\mathcal{A}$.

$$l(p) \leq (2n+1)+$$

$$5 \cdot 3^{\sum_{i=2}^{\infty} \frac{1}{F_i}} \cdot (k-2) \cdot (n+1)^{\frac{1}{F_{k-2}}} \cdot \left(\frac{n^2}{2}\right)^{1-\frac{1}{F_{k-2}}}.$$

$$\sum_{i=2}^{\infty} \frac{1}{F_i} \leq 1.43.$$

$$l(p) \leq 3n + 25 \cdot (k-2) \cdot \frac{2^{\frac{1}{F_{k-2}}}}{2^{1-\frac{1}{F_{k-2}}}} \cdot n^{\frac{1}{F_{k-2}}} \cdot n^{2-\frac{2}{F_{k-2}}}$$

$$\leq 25 \cdot k \cdot n^{2-\frac{1}{F_{k-2}}}.$$

□

Corollary 3 *There exists an absolute constant $C > 0$, so that if $k \leq C \log\log n$, then $L_k(n) = o(n^2)$.*

Proof. $F_k \leq 2^k$. $\square$